Computing the Capacity Region of a Wireless Network

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***Abstract*—We consider a wireless network of n nodes that communicate over a common wireless medium under some interference constraints. Our work is motivated by the need for an efficient and distributed algorithm to determine the n^2 dimensional unicast capacity region of such a wireless network. Equivalently, given a vector of end-to-end rates between various source-destination pairs, we seek to determine if it can be supported by the network through a combination of routing and scheduling decisions.**

This question is known to be NP-hard and hard to even approximate within $n^{1-o(1)}$ factor for general graphs. In this paper, we first show that the whole n^2 dimensional unicast capacity region can be approximated to $(1 \pm \varepsilon)$ factor in polynomial time, and in a distributed manner, whenever the Max Weight Independent Set (MWIS) problem can be approximated in a similar fashion for the corresponding topology. We then consider wireless networks which are usually formed between nodes that are placed in a geographic area and come endowed with a certain *geometry*, and argue that such situations do lead to approximations to the MWIS problem (in fact, in a completely distributed manner, in a time that is essentially linear in n). Consequently, this gives us a polynomial algorithm to approximate the capacity of wireless networks to arbitrary accuracy. This result hence, is in sharp contrast with previous works that provide algorithms with at least a constant factor loss. An important ingredient in establishing our result is the transient analysis of the maximum weight scheduling algorithm, which can be of interest in its own right.

I. Introduction

Wireless networks are becoming the architecture of choice for designing many of the emerging communication networks such as mesh networks to provide infrastructure in metro areas, peer-to-peer networks, and to provide infrastructure free interactions between handheld devices in popular locations like shopping malls or movie theatres, mobile ad-hoc network between vehicles for IVHS, etc. In all such settings, in essence we have a wireless network of n nodes where nodes are communicating over a common wireless medium using a certain standard communication protocol (e.g. IEEE 802.11 standard). Under any such protocol, transmission between a pair of nodes is successful iff none of the *nearby* or *interfering* nodes are transmitting simultaneously. Any such interference model is equivalent to an *independent set* interference model over the graph of interfering communication links.

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A key operational question in any such network (e.g. a mesh network in a metro area) is that of determining whether a given set of end-to-end rates between various source destination pairs is simultaneously supportable by the network. That is, one wishes to determine the n^2 dimensional unicast capacity region of such a wireless network of n nodes. An algorithm for determining this feasibility must be distributed (i.e. operation at a node utilizes only the information of the node's neighbors) and very efficient in order to be implementable.

However, an algorithm for determining feasibility of end-to-end rates has to explore over exponentially large space of joint routing and scheduling decisions under the wireless network interference constraints. This makes the question of designing such an efficient, distributed algorithm potentially very hard. Indeed, this question is known to be NP-hard and hard to approximation within n^δ factor (for some $\delta > 0$) for general graphs [12].

But since wireless networks are usually formed between nodes that are placed in a geographic area, they possess a natural *geometry*. Therefore, a natural question arises: is it possible to design efficient algorithms for checking end-to-end rate feasibility for a wireless network arising in practice (i.e. possessing geometry)?

A. Our contributions

As the main contribution of this paper we provide an answer to the above question by providing a polynomial algorithm to determine feasibility. Next, we describe various challenges faced, our approach and innovations.

From Single Hop to Multi hop Membership Oracles: In a previous work [11], we designed algorithms for single-hop feasibility for networks with geometry. In principle, a natural approach would be to make use of these single hop feasibility algorithm as an oracle repeatedly to derive an algorithm for checking feasibility of end-to-end rates for a multihop network. For the bounded density case, our algorithm provides ε-approximation in polynomial in n time and exponential in $1/\varepsilon$ for any $\varepsilon \in (0, 1)$. To use this as an oracle for end to end feasibility involves iterating over various routing choices along with the corresponding 'optimal' scheduling choices that are implicitly determined by the single hop oracle. Firstly, this would require an efficient algorithm to determine good routing choices, given the scheduling choices (the other way is implicitly given by the single hop rate feasibility algorithm), which is very non trivial. Further, another issue while invoking the single hop oracle multiple times is that it is a conditional approximation on the feasibility vector. We do note however that, in the case where nodes are restricted to one dimension (slab problem) which has a polynomial LP characterization in [11], as we show it trivially extends to solve the multi-hop membership problem exactly. This is discussed in section V towards the end.

Approach: In view of the above concerns, we take a different approach to directly tackle the multi-hop problem, which could be useful in a practical sense, provided the network graph allows for an efficient approximation to the MWIS problem. This approximate MWIS algorithm will be used towards a joint scheduling and routing algorithm over the interference communication network in a specific manner. A well known result of Tassiulas and Ephremides [13], says that if the given end-to-end rates are feasible, then a network with i.i.d. arrivals of mean equal to these rates (and of bounded second moment) under the maximum weight based combined scheduling and routing policy will lead to a stable Markov process of the queue lengths. This suggests the following vaguely stated approach for a feasibility test of end to end rates: Simulate a network with i.i.d. arrivals of means equal to the given end-to-end rates using an *approximate* MWIS algorithm. Hopefully, if the queues remain "**stable**" then the rates are *approximately* feasible or else they are *approximately* infeasible. This is the basic idea behind our approach. However, in order to make this approach 'feasible', we need to deal with a host of non-trivial issues that are stated below:

- Firstly, even if one had an efficient exact MWIS algorithm, its popular analysis (as in [14]) does not provide explicit absolute bounds on the queue lengths at any given time instant. This is because the bounds on queue-sizes are are only known existentially by the notion of stability and we need to characterize them explicitly in order to be able to design an algorithm.
- The queue length bounds obtained by using the standard Foster's criterion and moment bounds are only statements on the equilibrium distribution. This means that in order to establish an absolute bound, one might need to estimate the rates of convergence and we would have a polynomial algorithm if this convergence is quick enough. We take somewhat novel approach where we iterate analysis with the design along with the use of real valued queue lengths with deterministic fractional arrivals. By doing so, the queue-size vector does not remain a Markov chain on integer state space but, our direct analysis leads to bounds that are sufficient for our purposes and leads to the approximate correctness property of the algorithm that we propose.
- The MWIS is an NP-Hard problem even to approximate within $n^{1-o(1)}$ factor for general network topology. However, a wireless networks formed by nodes placed in a geographic area leads to a topology that has geometric properties. For such geometric wireless networks, the MWIS does have distributed and efficient **approximation** algorithms. We build upon such an approximate MWIS algorithm to get an algorithm that remains a good **approximation for end-to-end rate problem**.

B. Related work

In past decade or so, the emergence of wireless network architectures have led various researchers to take two different approaches to design efficient algorithms for checking feasibility of end-to-end rates.

The first approach is inspired by the possibility of deriving explicit simple bounds. Specifically, starting work by Gupta and Kumar [3] significant effort has been put in to derive simple scaling laws for large random wireless network for random traffic demands. In essence, this result implies that under such a random regular setup, per source supportable rate scales like $1/\sqrt{n}$ in the network of n nodes. Thus, if such a random setting is a good approximation of the network operating in practice, then one can utilize this $1/\sqrt{n}$ formula to determine approximate feasibility. The possible effectiveness of such an approach has led to an extensive study of a related notion of *transport capacity*, introduced in [3], over the past decade. For example, see works by [2], [5], [7] and many others. We also refer an interested reader to a comprehensive survey by Xue and Kumar [15]. More recently, a complete information theoretic characterization of random traffic demand were obtained for random node placement by Ozgur, Tse and Leveque [10] and for arbitrary node placement by Niesen, Gupta and Shah [9].

The second approach is based on determining the exact or approximate feasibility for a given arbitrary wireless network operating under a specific interference model. The question of determining feasibility of end-to-end rates is equivalent to checking feasibility of a solution of a certain Linear Program (LP). However, this LP is very high dimensional (due to exponentially many routing and scheduling choices) and hence exact solutions like simplex algorithm for this LP are inefficient. Various authors have provided approaches to design approximation algorithm with constant factor loss for such an LP with the constant factor loss being a function of the *degree* of nodes in the interference graph.

In general, given a specific network and a vector of end-to-end rates between various source destination pairs, there are no polynomial-time (approximation) algorithms to determine their feasibility. Of course, this is not feasible for an arbitrary graph as it is NP-hard. In [11], the problem of determining feasibility of rates when sources wishes to send data to their neighbors directly was considered, i.e. the problem of feasibility of rates for a single hop network. A polynomial time approximation was developed for this problem when the network possesses geometry. As explained earlier, the single hop rate feasibility algorithm does not lead to feasibility of end-to-end rate feasibility primarily due to additional freedom of routing over exponentially many choices. It is also important to note that the multi-hop routing version is not a generalization of the single hop problem. This is because, even if we consider all source destination pairs as 1-hop neighbors, it is possible that the rates are feasible through a multi-hop routing scheme while the trivial single hop routing itself is infeasible. In that sense, the two problems, though related, are in fact, not generalizations of one another.

C. Organization

The paper is organized as follows. In section II, we introduce the basic notation and define the problem, and state our basic result, which will be proved in section III by assuming an approximate MWIS algorithm. We then go on to describe the MWIS algorithm, and the restricted network graphs for which this can become feasible. Finally, in section V, we talk about network graphs where nodes are distributed in the plane while bounded in one dimension (with arbitrary second dimension), and extend the single hop rate feasibility to end to end feasibility.

II. Problem Statement and Main Result

Consider a wireless network on n nodes defined by a directed graph $G = (V, E)$ with $|V| = n, |E| = L$. For any $e \in E$, let $\alpha(e), \beta(e)$ denote respectively the origin and destination vertices of the edge e. The edges denote potential wireless links, and only the subsets of the edges that do not interfere can be simultaneously active. From now on, bold font indicates vectors or matrices. Let

$$\mathcal{S} = \{\mathbf{e} \in \{0,1\}^L : \mathbf{e} \text{ is the adjacency vector for a non-interfering subset of } E\} \quad (1)$$

Note that $\mathcal{S}$ is the collection of the *independent sets of E* by considering interference among $e_1 \in E$ and $e_2 \in E$ as the edge between them. Given a graph $G = (V, E)$ on n nodes and node weights given by $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{R}_+^n$, a subset $\mathbf{x}$ of V is said to be an *independent set* if no two vertices of $\mathbf{x}$ have common edge. let $\mathcal{I}(G)$ be set of all independent sets of G. A *maximum weight independent set* $\mathbf{x}^*$ is defined by $\mathbf{x}^* = \text{argmax}\left\{\mathbf{w}^T\mathbf{x} : \mathbf{x} \in \mathcal{I}(G)\right\}$, where we consider $\mathbf{w}$ as an element of $\{0,1\}^{|V|}$. Given $\varepsilon > 0$, a subset $\hat{\mathbf{w}} \in \{0,1\}^{|V|}$ is called an *ε-approximation of MWIS* if $\hat{\mathbf{x}} \geq (1-\varepsilon)\mathbf{w}^T\mathbf{x}^*$.

The convex hull of $\mathcal{S}$, denoted by $co(\mathcal{S})$ in $\mathbb{R}_+^L$ represents the link rate feasibility region. Typically, the set $co(\mathcal{S})$ is complicated to describe (and exponential in size). Determining membership in $co(\mathcal{S})$ was a problem shown to be NP-hard by [1] under general interference constraints. For restricted (node exclusive) interference constraints, and general graphs, [4] exhibits polynomial algorithms. For general interference models, but some restricted networks, polynomial algorithms were given in [11].

In this paper, we consider m distinct source destination pairs, $(s_1, d_1), \dots, (s_m, d_m)$ and an *end to end* rate vector, $\mathbf{r} = (r_1, r_2, \dots, r_m) \in [0,1]^m$. We will usually use index j to range over the S-D pairs in the following discourse.

Definition 1: The rate vector $\mathbf{r} = (r_1, \dots, r_m) \in [0,1]^m$ corresponding to the S-D pairs $(s_1, d_1), \dots, (s_m, d_m)$ is said to be '**feasible**', if there exist flows, $(\mathbf{f}^1, \dots, \mathbf{f}^m)$ such that

- $\mathbf{f}^j$ routes a flow of at least r_j from s_j to d_j for $1 \leq j \leq m$
- The net flow on the links induced, $\hat{\mathbf{f}} \doteq \sum_{j=1}^m \mathbf{f}^j$ belongs to $co(\mathcal{S})$, i.e. in other words, it can be scheduled under the interference constraints with a schedule of at most unit length.

The equations that specify the notion of "flows routing $\mathbf{r}$" are given later in (10) via (2) and (8). Let

$$\mathcal{F} = \{\mathbf{r} \in [0,1]^m : \mathbf{r} \text{ is 'feasible'}\}$$

be the set of all feasible end to end rate vectors. Our primary result is the following.

Theorem 1: Assume we have an $1-\varepsilon$- approximation algorithm to determine the Max Weight independent set of a class of wireless networks for $0 < \varepsilon < 1/4$. Then, there exists a deterministic polynomial time algorithm to determine the approximate rate feasibility of a given end to end rate vector $\mathbf{r}$ in the following sense: If $(1+2\varepsilon)\mathbf{r} \in \mathcal{F}$, then the algorithm outputs a 'YES'. Conversely, if $(1-2\varepsilon)\mathbf{r} \notin \mathcal{F}$, then the algorithm outputs a 'NO'. Else, the answer could be arbitrary.

We note that, the only restrictions on the graph structure assumed arise from the requirements for MWIS approximation. Hence, given any general network where the MWIS can be approximated, the result can be exploited in that framework.

III. Proof of Theorem 1

To prove Theorem 1, we describe the algorithm first with some parameters the algorithm uses to compute its answer. Let t be an index ranging over integers, to be interpreted as slotted time. Define $q_i^j(t) \in \mathbb{R}_+$ as the 'packet mass' at node i destined for node d_j at time t (for $1 \leq i \leq n, 1 \leq j \leq m$). Define m 'routing matrices', each of dimension $n \times L$ with the j^{th} matrix, $\mathbf{R}^j$ defined as follows via its $(i,l)^{th}$ element ($1 \leq i \leq n$, and $1 \leq l \leq L$):

$$R_{i,l}^j = \begin{cases} -1 & \text{if } \alpha(l) = i, d_j \neq i \\ 1 & \text{if } \beta(l) = i, d_j \neq i \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Define a 'weight matrix' at time t, $\mathcal{W}(t)$, of dimension $L \times m$ via its $(l,j)^{th}$ element ($1 \leq l \leq L$ and $1 \leq j \leq m$):

$$\mathcal{W}_l^j(t) = q_{\alpha(l)}^j(t) - q_{\beta(l)}^j(t). \quad (3)$$

In vector notation[1], we have for $1 \leq j \leq m$:

$$-\mathcal{W}^j(t)^T = \mathbf{q}^j(t)^T\mathbf{R}^j, \quad (4)$$

The weight vector of dimension L, $\mathbf{W}(t)$, is then defined with its l^{th} element (corresponding to link $l, 1 \leq l \leq L$) as

$$W_l(t) = \max_j\{\mathcal{W}_l^j(t)\}. \quad (5)$$

Finally, let the Maximum weight of the non interfering set of links be [2]

[1]Omitting a subscript for a previously defined scalar represents the corresponding column vector.

[2]**a**.**b** denotes the standard vector dot product of **a** and **b**.

$$\mathcal{M}(t) = \max_{\mathbf{e}\in\mathcal{S}} \mathbf{e}.\mathbf{W}(t). \tag{6}$$

Property 2: $\varepsilon-$ MWIS returns some $\hat{\mathbf{e}}(t) \in \mathcal{S}$ with the following property for each given t:

$$\hat{\mathbf{e}}(t).\mathbf{W}(t) \geq (1-\varepsilon)\mathcal{M}(t). \tag{7}$$

The 'link activation matrix', $\mathcal{E}(t) \in \{0,1\}^{Lm}$, of dimension $L \times m$, will now be defined using the vector $\hat{\mathbf{e}}(t)$ obtained above for a fixed t. The $(l,j)^{th}$ element, $\mathbf{E}_l^j(t) = 1$ is to be interpreted as activating link l to transfer a unit packet mass corresponding to S-D pair j at the beginning of time slot $t+1$. Note that the MWIS approximation algorithm itself is oblivious to the various types of packets in the networks. So, we need to convert the set $\hat{\mathbf{e}}(t)$ into specific information on which class of S-D pair packets that it needs to serve, which will be accomplished while defining the link activation matrix below.

Definition 2 (Link Activation Matrix): We write $\mathcal{E}(t) = [\mathbf{E}^1(t)\ldots\mathbf{E}^m(t)]$, and define the columns, $\mathbf{E}^j$'s in what follows. For $1 \leq j \leq m$, let:

$$S^j = \{l : \hat{e}_l(t) = 1, W_l(t) = W_l^j(t) \text{ and } W_l^{j'}(t) < W_l(t), \forall j' < j\}.$$

S^j's are all disjoint sets by definition and $\cup_{j=1}^m S^j$ is a subset of E with adjacency vector $\hat{\mathbf{e}}(t)$. $\mathbf{E}^j(t)$ is then defined to be the adjacency vector of some **maximal subset of S^j that can be activated, subject to the following constraint**:

Property 3 (activation constraint): The total number of activated links pointing out of node i in the activation set represented by $\mathbf{E}^j(t)$ is at most $q_i^j(t)$ for $1 \leq i \leq n$.

Remark 4: The above constraint is included to ensure that the queue sizes do not become negative because of activating too many links while having too less queue size at any given node. Because of this, the MWIS algorithm is supplied with positive weights, and the analysis below can assume that $q_i^j(t) \geq 0$, $\forall t$.

Let $\mathbf{E}(t)$ be the net activation vector: $\mathbf{E}(t) = \sum_{j=1}^m \mathbf{E}^j(t)$. It has the following property:

Property 5: $\mathbf{W}(t).\mathbf{E}(t) \geq (1-\varepsilon)\mathcal{M}(t) - n^3$.

Proof: $\mathbf{W}(t).\mathbf{E}(t) = \mathbf{W}(t).\hat{\mathbf{e}}(t) - \mathbf{W}(t).(\hat{\mathbf{e}}(t) - \mathbf{E}(t)) \geq (1-\varepsilon)\mathcal{M}(t) - nL$ (the bound for the first term follows from property 2, and for the second term since for any $1 \leq l \leq L$, $\hat{e}_l(t) - E_l(t) = 1 \Rightarrow W_l(t) \leq n$ based on the activation constraint above.) ∎

We now describe the actual queue computations performed by the algorithm. We would like to make the observation that all these operations can be performed in a completely distributed fashion with simple local updates. provided that the MWIS can be implemented in a distributed manner, which will be described later in section IV.

Queue computations done by the algorithm:

The algorithm simulates the following steps on the given network model.

- Initialize all the mn queues, $q_i^j, 1 \leq i \leq n, 1 \leq j \leq m$ to zero mass at $t = 0$. Subsequently, at each discrete time slot, do the following:

(1) Add (a 'packet mass' of) r_k to $q_{s_k}^k$.
(2) Compute the weight matrix, $\mathbf{W}(t)$ via equations 3, 5.
(3) Invoke the $\varepsilon-$MWIS algorithm with weights $\mathbf{W}(t)$, which results in $\hat{\mathbf{e}}(t)$ satisfying Property 2.
(4) Decide the link activation matrix, $\mathcal{E}(t) \in \{0,1\}^{Lm}$ using the specification in definition 2 (such that it satisfies Property 3)
(5) For each activated link, l, with $\mathbf{E}_l^j(t) = 1$, move a unit queue mass from $q_{\alpha(l)}^j$ to $q_{\beta(l)}^j$. In other words, make the following updates:

$$q_{\alpha(l)}^j(t+1) = q_{\alpha(l)}^j(t+1) - 1$$

and

$$q_{\beta(l)}^j(t+1) = q_{\beta(l)}^j(t) + 1$$

We'll now model the process specified above. Towards this, define m 'arrival vectors', $\mathbf{a}^j$ for $1 \leq j \leq m$, each of dimension L as (corresponding to step 1):

$$a_i^j = \begin{cases} r_j & \text{if } i = s_j \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

The queue dynamics then follows:

$$\mathbf{q}^j(t+1) = \mathbf{q}^j(t) + \mathbf{R}^j\mathbf{E}^j(t) + \mathbf{a}^j. \tag{9}$$

Let, the maximum queue size observed across the network at time t be:

$$q^{max}(t) \triangleq \max_{(i,j)} q_i^j(t).$$

We can then prove the following lemma:

Lemma 6: If $(1+2\varepsilon)\mathbf{r} \in \mathcal{F}$, then $q^{max}(t) \leq \frac{10}{\varepsilon}n^{7.5}$ for all $t > 0$ and $0 < \varepsilon < 1/4$.

Proof: Consider the standard quadratic potential function $V(t) = \sum_{i,j}(q_i^j(t))^2$. Then,

$$\begin{aligned}
&\Delta V(t) = V(t+1) - V(t) \\
&= \sum_j (\mathbf{R}^j\mathbf{E}^j(t) + \mathbf{a}^j).(\mathbf{R}^j\mathbf{E}^j(t) + \mathbf{a}^j + 2\mathbf{q}^j(t)) \\
&= \sum_j (\mathbf{R}^j\mathbf{E}^j(t) + \mathbf{a}^j).(\mathbf{R}^j\mathbf{E}^j(t) + \mathbf{a}^j) \\
&+2\sum_j \mathbf{q}^j(t).(\mathbf{R}^j\mathbf{E}^j(t) + \mathbf{a}^j) \\
&\leq (n+1)^2 m + 2\left(\sum_j \mathbf{q}^j(t).\mathbf{R}^j\mathbf{E}^j(t) + \sum_j \mathbf{q}^j(t).\mathbf{a}^j\right) \\
&\leq 2n^4 + 2(A+B)
\end{aligned}$$

where, we now bound the terms A and B, starting with A first:

$$
\begin{aligned}
A = \sum_{j=1}^{m} (\mathbf{q}^j(t))^T \mathbf{R}^j \mathbf{E}^j(t) \;&=\; -\sum_{j=1}^{m} \mathcal{W}^j(t)^T \mathbf{E}^j(t) \\
= -\sum_{j=1}^{m} \mathbf{W}(t)^T \mathbf{E}^j(t) \quad &\left(\because \mathbf{E}_l^j \neq 0 \Rightarrow \mathbf{W}_l = \mathcal{W}_l^j\right) \\
= -\,\mathbf{W}(t).\mathbf{E}(t) \leq -(1-\varepsilon)\mathcal{M}(t) + n^3. \;&(\text{ from Prop 5})
\end{aligned}
$$

Now we bound term B. Note that a 'flow vector', $\mathbf{f}^j \in \mathbb{R}_+^L$ routes a flow r_j for the S-D pair j, if the following holds:

$$
\mathbf{a}^j = -\mathbf{R}^j \mathbf{f}^j \text{ for } 1 \leq j \leq m. \tag{10}
$$

Let $(\mathbf{f}^1, \ldots \mathbf{f}^m)$ 'route' $\mathbf{r}$ for the m S-D pairs. The net flow on the links is given by:

$$
\hat{\mathbf{f}} = \sum_{j=1}^{m} \mathbf{f}^j.
$$

Claim 7: If $(1+2\varepsilon)\mathbf{r} \in \mathcal{F}$, then there exist flows $(\mathbf{f}^1, \ldots, \mathbf{f}^m)$ that route $\mathbf{r}$ such that for the net flow, $\hat{\mathbf{f}}$, the following relation holds: $(1+2\varepsilon)\hat{\mathbf{f}} \in co(\mathcal{S})$.

Proof: Since, $(1+2\varepsilon)\mathbf{r} \in \mathcal{F}$, there exist flows $(\mathbf{g}^1, \ldots \mathbf{g}^m)$ that route $(1+2\varepsilon)\mathbf{r}$ with $\hat{\mathbf{g}} \in co(S)$. Define $\mathbf{f}^j = \frac{1}{1+2\varepsilon}\mathbf{g}^j$. Since $\mathbf{a}^j$'s are linear in $\mathbf{r}$ (eq. 8), we see (from eq. 10) that, if $(\mathbf{g}^1, \ldots \mathbf{g}^m)$ route $(1+2\varepsilon)\mathbf{r}$, then $(\mathbf{f}^1, \ldots \mathbf{f}^m) = \frac{1}{1+2\varepsilon}(\mathbf{g}^1, \ldots \mathbf{g}^m)$ route $\mathbf{r}$. Also, from linearity of $\hat{\mathbf{f}}$ in terms of $(\mathbf{f}^1, \ldots, \mathbf{f}^m)$, it follows that $(1+2\varepsilon)\hat{\mathbf{f}} = \hat{\mathbf{g}} \in co(\mathcal{S})$. ∎

Now, assuming that $(1+2\varepsilon)\mathbf{r} \in \mathcal{F}$,

$$
\begin{aligned}
B &= \sum_j \mathbf{q}^j(t).\mathbf{a}^j \\
&= -\sum_j (\mathbf{q}^j(t))^T \mathbf{R}^j \mathbf{f}^j \left(\text{ where } (1+2\varepsilon)\hat{\mathbf{f}} \in co(\mathcal{S})\right) \\
&= \sum_j \mathcal{W}^j(t)^T \mathbf{f}^j \leq \mathbf{W}(t)^T \sum_j \mathbf{f}^j = \mathbf{W}(t).\hat{\mathbf{f}}.
\end{aligned}
$$

Since $(1+2\varepsilon)\hat{\mathbf{f}} \in co(\mathcal{S})$, let

$$
(1+2\varepsilon)\hat{\mathbf{f}} = \sum_{i=1}^{|\mathcal{S}|} \lambda_i \mathbf{c}_i \text{ where each } \mathbf{c}_i \in \mathcal{S} \tag{11}
$$

for some non negative λ_i such that $\sum_i \lambda_i \leq 1$. Then,

$$
\begin{aligned}
B &\leq \frac{1}{1+2\varepsilon} \sum_{i=1}^{|\mathcal{S}|} \lambda_i \mathbf{W}(t).\mathbf{c}_i \\
&\leq \frac{1}{1+2\varepsilon} \mathcal{M}(t).
\end{aligned}
$$

Therefore, we have the following bound on $\Delta V(t)$:

$$
\begin{aligned}
\Delta V(t) &\leq 2n^4 + 2\left(n^3 + \mathcal{M}(t)\left(\frac{1}{1+2\varepsilon} - (1-\varepsilon)\right)\right) \\
&\leq 3n^4 - \frac{\varepsilon}{3}\mathcal{M}(t) \text{ for } 0 < \varepsilon < 1/4.
\end{aligned}
$$

Next note that we could assume without loss of generality that the network graph is connected. If this is not the case, we can always analyze the capacity regions of each connected component separately. Upon this, we can get the simple bound that:

$$
\mathcal{M}(t) \geq \frac{1}{L}\sqrt{\frac{V(t)}{mn}} \geq \frac{\sqrt{V(t)}}{n^{3.5}}.
$$

To see the above bound, let $(a,b) = \arg max_{(i,j)} q_i^j$. Then $q_a^b \geq \sqrt{\frac{V(t)}{mn}}$ and since the network is connected, on any path from a to d_b (of length at most L), there exists at least one link, l such that $\mathcal{W}_l^j(t) = q_{\beta(l)}^j(t) - q_{\alpha(l)}^j(t) \geq \frac{1}{L} q_a^b \geq \frac{1}{L}\sqrt{\frac{V(t)}{mn}}$, which is clearly a lower bound for $\mathcal{M}(t)$.

Therefore, we have the following bound

$$
\Delta V(t) \leq 3n^4 - \frac{\varepsilon}{3}\frac{\sqrt{V(t)}}{n^{3.5}} \tag{12}
$$

which in turn implies that

$$
V(t) \leq 3n^4 + \left(\frac{9n^{7.5}}{\varepsilon}\right)^2 \leq \frac{100}{\varepsilon^2} n^{15} \quad \forall t > 0.
$$

Hence,

$$
\max_{(i,j)} q_i^j(t) \leq \frac{10}{\varepsilon} n^{7.5} \quad \forall t > 0.
$$

∎

Lemma 8: If $(1-2\varepsilon)\mathbf{r} \notin \mathcal{F}$, then $q^{max}(t) \geq \frac{\varepsilon^2}{n^4} t$ for all $t \geq 0$.

Proof: First, we begin by showing the following claim:

Claim 9: if $(1-2\varepsilon)\mathbf{r} \notin \mathcal{F}$, then for any given m link rate vectors, $(\mathbf{g}^1, \ldots \mathbf{g}^m)$ with $\hat{\mathbf{g}} := \sum_j \mathbf{g}^j \in co(\mathcal{S})$, there is $j \in [m]$ such that the followings hold:

(a) The graph with edge capacities given by $\mathbf{g}^j$ has a maximum flow of value at most $(1-\varepsilon)r_j$ from s_j to d_j.
(b) $r_j > \frac{\varepsilon}{n^3}$.

Proof: Suppose that Claim 9 is not true. Then, let

$$
\mathbf{i} = \{i \in [m] | r_i \geq \frac{\varepsilon}{n^3}\}.
$$

By the assumption, for all $i \in \mathbf{i}$, there exists a flow of value at least $(1-\varepsilon)r_i$ from s_i to d_i for edge capacities defined according to $\mathbf{g}^i$. Now consider m link rate vectors, $(\mathbf{h}^1, \ldots \mathbf{h}^m)$ with $\mathbf{h}^i \in \mathbb{R}_+^L$ defined below. Let R_i be some fixed arbitrary path from s_i to d_i:

$$\mathbf{h}_l^i = \begin{cases} (1-\varepsilon)\mathbf{g}_l^i & \text{if } i \in \mathbf{i}, l \in E \\ \frac{\varepsilon}{n^3} & \text{if } i \notin \mathbf{i} \text{ and } l \in R_i \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

That is, for $i \in \mathbf{i}$, $\mathbf{h}^i = (1-\varepsilon)\mathbf{g}^i$ and otherwise, the value of $\mathbf{h}^i$ on all the edges in the path R_i is equal to $\frac{\varepsilon}{n^3}$, while the value on any edge not in R_i is defined to be 0. We will now argue that the net link rate, $\hat{\mathbf{h}} = \sum_j \mathbf{h}^j \in co(S)$ by producing a schedule for $\hat{\mathbf{h}}$ of unit length.

Note that for $i \in [m] \cap \mathbf{i}^c$, $\mathbf{h}^i$ can be scheduled under any interference constraint using $\frac{\varepsilon}{n^3} \times n$ amount of time, as there are at most n links in the path R_i. Thus, the link rate vector, $\sum_{i \in [m] \cap \mathbf{i}^c} \mathbf{h}^i$ can be scheduled in at most $m \times \frac{\varepsilon}{n^2} \leq \varepsilon$ time.

Next, $\sum_{i \in \mathbf{i}} \mathbf{h}^i = (1-\varepsilon)\sum_{i\in\mathbf{i}} \mathbf{g}^i \leq (1-\varepsilon)\sum_{i\in[m]} \mathbf{g}^i = (1-\varepsilon)\hat{\mathbf{g}}$. Since, $\hat{\mathbf{g}} \in co(\mathcal{S})$, this implies that $\sum_{i\in\mathbf{i}} \mathbf{h}^i$ can be scheduled in a total of $(1-\varepsilon)$ time.

Hence, $\hat{\mathbf{h}} = \sum_{i\in[m]} \mathbf{h}^i = \sum_{i\in\mathbf{i}} \mathbf{h}^i + \sum_{i\in[m]\cap\mathbf{i}^c} \mathbf{h}^i$ can be scheduled in $(1-\varepsilon)+\varepsilon$, which is unit time and thus we have $\hat{\mathbf{h}} \in co(\mathcal{S})$.

Now, consider the graph with edge capacities $\mathbf{h}^i$. If $i \in \mathbf{i}$, the max flow from s_i to d_i is at least $(1-\varepsilon)^2 r_i \geq (1-2\varepsilon)r_i$ by the definition of $\mathbf{h}^i$. Else, if $i \in [m] \cap \mathbf{i}^c$, then the max flow is again at least $\frac{\varepsilon}{n^3}$, which is bigger than $(1-2\varepsilon)r_i$ by the definition of $\mathbf{i}$. This implies that for each $i \in [m]$, there exists a flow ϕ^i which routes at least $(1-2\varepsilon)r_i$ from s_i to d_i, while satisfying (componentwise), $\phi^i \leq \mathbf{h}^i$. Thus, the vector of flows, $(\phi_1, \ldots, \phi_m)$ routes $(1-2\varepsilon)\mathbf{r}$ with the net flow, $\hat{\phi} \leq \hat{\mathbf{h}} \in co(\mathcal{S})$ Hence, $(1-2\varepsilon)\mathbf{r} \in \mathcal{F}$, contradicting our assumption and we obtain Claim 9. ■

Now, for any $t > 0$, define $(\mathbf{g}^1(t), \ldots \mathbf{g}^m(t))$, as the link rates for each packet type obtained by considering the actual schedules. More precisely, For $j \in [m]$, define

$$\mathbf{g}_l^j(t) = \frac{1}{t}(|\{t : \mathbf{E}_l^j(t) = 1\}|) \tag{14}$$

Clearly, $\hat{\mathbf{g}}(t) \in co(\mathcal{S})$, and we can apply Claim 9 to it. Thus, there exists $j \in [m]$ such that $r_j > \frac{\varepsilon}{n^3}$ and the max flow from s_j to d_j is at most $(1-\varepsilon)r_j$ in the graph with edge capacities $\mathbf{g}^j(t)$. Applying the max-flow min-cut theorem, observe that there is a cut (S,T) of the vertex set V such that $s_j \in S$, $t_j \in T$ and $c(S,T) = \sum_{l\in E:\alpha(l)\in S, \beta(l)\in T} \mathbf{g}_l^j(t)$ is equal to the max flow from s_j to t_j, which is at most $(1-\varepsilon)r_j$.

From equation 14, observe that the total amount of packet mass of type j that was moved from S to T during during the t time slots is at most $\sum_{l\in E:\alpha(l)\in S,\beta(l)\in T} t \times \mathbf{g}_l^j(t)$, which is at most $t \times (1-\varepsilon)r_j$. Since the amount of packets of type j that were added during these time slots is $t \times r_i$, we have:

$$\sum_{i\in S} q_i^j(t) \geq t.r_j - t.(1-\varepsilon)r_j.$$

Therefore,

$$\max_{(i,j)} q_i^j(t) \geq \frac{\sum_{i\in S} q_i^j(t)}{|S|} \geq \frac{\varepsilon \cdot t \cdot r_j}{n} \geq \frac{\varepsilon^2}{n^4} t.$$

■

Theorem 1 is now a consequence of Lemmas 6 and 8 since since the max queue size grows at least linearly with time if $(1-2\varepsilon)\mathbf{r} \notin \mathcal{F}$ and is polynomially bounded if $(1+2\varepsilon)\mathbf{r} \in \mathcal{F}$, so the two cases can be clearly distinguished in the worst case before $t = \frac{10}{\varepsilon^3} n^{11.5}$ time slots of simulation. Note that the Maximum queue size can be spread across the network in a distributed manner easily. Further, the queue computation updates are also essentially distributed computations.

A. Practical Implications for a Capacity membership Test

Combining the above lemmas, we have the following results based on the Maximum queue size observed at each time upon simulating the virtual queue computations using approximate max weight scheduling and routing described.

Let event E_1 be defined as observing

$$q^{max}(t) > \frac{10}{\varepsilon} n^{7.5} \text{ for some } t > 0.$$

Similarly, define event E_2 as observing

$$q^{max}(t) < \frac{\varepsilon^2}{n^4} t \text{ for some } t > 0.$$

We run the algorithm till a time $\hat{T}$ where:

$$\hat{T} = \min_t \{E_1 \text{ or } E_2 \text{ occurs}\}.$$

Note that either E_1 or E_2 has to occur eventually (in the worst case, before $t = \frac{10}{\varepsilon} n^{11.5}$ by definition of E_1 and E_2, so $\hat{T}$ is clearly polynomial)

We can then declare the following $2\varepsilon-$approximate statements (for arbitrarily small $\varepsilon > 0$,) on the membership of $\mathbf{r}$ in $\mathcal{F}$ by observing $q^{max}(\hat{T})$.

1) If E_1, then declare $(1+2\varepsilon)\mathbf{r} \notin \mathcal{F}$
2) If E_2, then declare $(1-2\varepsilon)\mathbf{r} \in \mathcal{F}$

The consistency of the above statements is a direct consequence of the definitions of E_1, E_2 and $\hat{T}$. Note that it is also possible that both E_1 and E_2 hold simultaneously without any contradiction, which just means that $\mathbf{r}$ is within an $1 \pm 2\varepsilon$ factor close to the boundary of the capacity region.

Alternately, one may not have any ε pre-specified to begin with and the interest is simply in making the best possible approximate statement after running the algorithm for a certain amount of time. We also have such a possibility resulting from the above analysis:

Towards this, define:

$$\varepsilon(t) = 2\min\left(n^2\sqrt{\frac{q^{max}(t)}{t}}, \frac{10n^{7.5}}{q^{max}(t)}\right)$$

Then, the above discussion implies that whenever $\varepsilon(t) < 1/2$, one can correctly declare the feasibility of a rate vector that is a $1 \pm \varepsilon(t)$ factor of the given vector. Further, given any $\varepsilon > 0$, we will have $\varepsilon(t) < \varepsilon$ for a polynomially bounded t.

B. Numerical Experiment

We simulated the algorithm on a directed cyclical network of 10 nodes shown in Figure 1 with 2-hop interference constraints by using standard software for solving Integer Linear Programs for the approximate MWIS. There were assumed to be 4 flows in contention. We plot the maximum queue size over time in Figure 2.

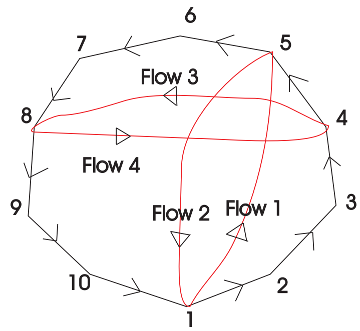

Fig. 1. An illustration of the cyclical network with 2-hop interference on which the algorithm was run. 4 dimensional rate vectors with coordinates corresponding to flows between nodes $1 \to 5, 5 \to 1, 4 \to 8, 8 \to 4$ were considered

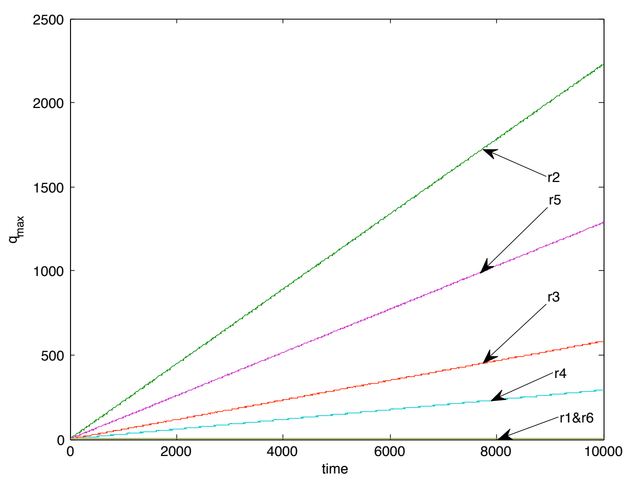

Fig. 2. A plot of $q^{max}(t)$ versus t on the network in Figure 1 for 6 different rate vectors given by $r_1 = [0.1\ \ 0.1\ \ 0.1\ \ 0.1], r_2 = [0.5\ \ 0.5\ \ 0.5\ \ 0.5], r_3 = [0.1\ \ 0.2\ \ 0.3\ \ 0.4], r_4 = [0.3\ \ 0.0\ \ 0.3\ \ 0.0], r_5 = [0.5\ \ 0.0\ \ 0.0\ \ 0.5], r_6 = [0.2\ \ 0.0\ \ 0.2\ \ 0.0]$. As we can see, q^{max} grows roughly linearly for r_2, r_3, r_4, r_5 whereas it stabilizes fairly quickly for r_1 and r_6. While our proofs give precise bounds and guarantees regarding polynomial convergence, these experimental plots suggest that in practice they are likely to be distinguished fairly quickly at least for simple topologies.

IV. ε-APPROXIMATION OF MWIS

In this section, we will present the definition and some properties of *polynomially growing graphs*, and an ε-approximation of MWIS for polynomially growing graphs. Let $\mathbf{d}_G$ be the shortest path distance of G and let $\mathbf{B}_G(v, r) = \{w \in V | \mathbf{d}_G(w, v) \le r\}$.

Definition 3: Given a graph G, if there are constants $C > 0$ and $\rho > 0$ so that for any $v \in V$ and $r \in \mathbb{N}$,

$$|\mathbf{B}_G(v, r)| \le C \cdot r^\rho,$$

then we say G is *polynomially growing*. Smallest such $\rho > 0$ is called the *growth rate* of G.

Many classes of network graphs arising in practice, including the following class of geometric network graph $G = (V, E)$, have polynomially growing property. Consider a wireless network of n nodes represented by $V = \{1, \dots, n\}$ placed in a 2-dimensional geographic region in an *arbitrary* manner (not necessarily random) inside a $\sqrt{n} \times \sqrt{n}$ square of area n.[3] Let $E = \{(i, j) : i$ can transmit to $j\}$ be the set of directed links between nodes indicating which nodes can communicate. We assume that the wireless network satisfies the following simple assumptions. Let $\mathbf{d}(\cdot, \cdot)$ be the Euclidean distance. Given a vertex $v \in V$, let $\mathbf{B}(v, R) = \{u \in V : \mathbf{d}(u, v) \le R\}$.

- There is an $R > 0$ such that no two nodes having distance larger than R can establish a communication link with each other[4] where R is bound on transmission radius.
- Graph G has bounded density $D > 0$, i.e. for all $v \in V$, $\frac{|\mathbf{B}(v,R)|}{R^2} \le D$.

A geometric random graph obtained by placing n nodes in the $\sqrt{n} \times \sqrt{n}$ square uniformly at random and connecting any two nodes that are within distance $r = \Theta(\sqrt{\log n})$ of each other satisfies the previous assumptions with high probability [11].

Lemma 10: Any geometric graph G with bounded density and bounded transmission radius R has growth rate 2.

Proof: First, note that in the Euclidean space, since $\mathbf{B}(v, r)$ can be covered by $\Theta\left(\left(\frac{r}{R}\right)^2\right)$ many balls of of radius R, there is a constant D' so that for all $v \in V$ and $r > 0$, $\frac{|\mathbf{B}(v,r)|}{r^2} \le D'$.

For two vertices $v, w \in V$, let $v = v_0, v_1, v_2 \dots, v_\ell = w$ be the shortest path in G. Then, by the bounded transmission radius property, for all $i = 0, 1 \dots, (\ell - 1)$, $\mathbf{d}(v_i, v_{i+1}) \le R$. By the triangular inequality in the Euclidean metric,

$$\mathbf{d}(v, w) \le \sum_{i=0}^{\ell} \mathbf{d}(v_i, v_{i+1}) \le R\ell.$$

So, $\mathbf{d}(v, w) \le R\ell = R\mathbf{d}_G(v, w)$. Hence, for any $v \in \hat{V}$ and $r \in \mathbb{N}$, $\mathbf{B}_G(v, r) \subset \mathbf{B}_G(v, Rr)$, which implies that

$$|\mathbf{B}_G(v, r)| \le |\mathbf{B}(v, Rr)| \le \left(D'R^2\right) r^2.$$

■

Lemma 11: If G is polynomially growing with growth rate ρ, any subgraph $\hat{G} = (\hat{V}, \hat{E})$ of G obtained by removing some edges and vertices of G is also polynomially growing with growth rate at most ρ.

Proof: For any vertex $v, w \in \hat{V}$, note that $\mathbf{d}_{\hat{G}}(v, w) \ge \mathbf{d}_G(v, w)$, since any path in $\hat{G}$ from v to w is also a path in G. Hence, for any $v \in \hat{V}$ and $r \in \mathbb{N}$, $\mathbf{B}_{\hat{G}}(v, r) \subset \mathbf{B}_G(v, r)$, which shows the lemma from the definition 3. ■

[3] Placing the nodes in the specified square is for simple presentation. The same result holds when the nodes are placed in any Euclidean rectangle, and when the nodes are place in k-dimensional Euclidean space.

[4] It does not imply that nodes within distance R must communicate.

Given a graph $G = (V, E)$, and a constant $\ell \in \mathbb{N}$, ℓ-depth adjacency graph $\bar{G} = (\bar{V}, \bar{E})$ of G is defined as follows. Let $\bar{V} = E$, and for any $\bar{e}_1, \bar{e}_2 \in \bar{V}$, $(\bar{e}_1, \bar{e}_2) \in \bar{E}$ iff an end point of $\bar{e}_1$ and an end point of $\bar{e}_2$ (as an edge of G) has shortest distance at most $(\ell - 1)$ in G. Notice that in our rate feasibility check algorithm, given a network graph G, we run an ε-approximate MWIS on a subgraph of the ℓ-depth adjacency graph of G.

Lemma 12: If G is polynomially growing with growth rate ρ, then for any constant $\ell \in \mathbb{N}$, its ℓ-depth adjacency graph is polynomially growing with growth rate 2ρ.

Proof: Fix $\bar{e}_0 \in \bar{V}$ and $r > 0$. Let $v_1, v_2 \in V$ be the two end points of $\bar{e}_0$ in G. Then, for all $\bar{e} \in \mathbf{B}_{\bar{G}}(\bar{e}_0, r)$, the two end points of $\bar{e}$ must belong to $\mathbf{B}_G(v_1, \ell r) \cup \mathbf{B}_G(v_2, \ell r)$. Hence, we obtain that

$$\begin{aligned} |\mathbf{B}_{\bar{G}}(\bar{e}, r)| &\leq \binom{|\mathbf{B}_G(v_1, \ell r) \cup \mathbf{B}_G(v_2, \ell r)|}{2} \\ &\leq \binom{2C(\ell r)^\rho}{2} \leq \left(4C^2 \ell^{2\rho}\right) r^{2\rho}. \end{aligned}$$

■

From Lemma 11 and Lemma 12, if the network graph G itself is polynomially growing, then edge interference graph of G is also polynomially growing.

In [6], Jung and Shah presented an ε-approximation algorithm for MWIS for any graph with polynomial growth ρ, which runs in time $O(n)$ for any constant $\varepsilon > 0$ and $\rho > 0$. We present the algorithm for completeness. The algorithm consists of the following steps.

1) Obtain a graph decomposition of G into small components by removing some vertices of G.
2) Compute optimal solutions locally in each of these components.
3) Produce a global solution by merging the local solutions.

To explain the first step, given $\varepsilon > 0$ and a constant $\Delta > 0$, we first define the notion of (ε, Δ)-decomposition for a graph $G = (V, E)$.

Definition 4: We call a random subset of vertices $\mathcal{B} \subset V$ as (ε, Δ)-decomposition of G if the followings hold:

1) For any $v \in V$, $\mathbb{P}(v \in \mathcal{B}) \leq \varepsilon$.
2) Let $S_1, \ldots, S_\ell$ be the connected components of graph $G' = (V', E')$ where $V' = V \backslash \mathcal{B}$ and $E' = \{(u, v) \in E : u, v \in V'\}$. Then, $\max_{1 \leq k \leq K} |S_k| \leq \Delta$ with probability 1.

Now we describe a graph decomposition algorithm for any $\varepsilon > 0$ and an operational parameter K. The algorithm outputs (ε, Δ)-decomposition where Δ will depend on K and ρ [6].

Given ε and K, define random variable $\mathbf{Q}$ over $\{1, \ldots, K\}$ as

$$\mathbb{P}[\mathbf{Q} = i] = \begin{cases} \varepsilon(1-\varepsilon)^{i-1} & \text{if } 1 \leq i < K \\ (1-\varepsilon)^{K-1} & \text{if } i = K \end{cases}.$$

Set

$$K = \frac{24\rho}{\varepsilon} \log\left(\frac{48\rho}{\varepsilon}\right).$$

Decomposition Algorithm (ε, K)

(1) Initially, set $\mathcal{W} = V$, $\mathcal{B} = \emptyset$ and $\mathcal{R} = \emptyset$.
(2) Repeat the following till $\mathcal{W} \neq \emptyset$:
   (a) Choose an element $u \in \mathcal{W}$ uniformly at random.
   (b) Draw a random number Q independently according to the distribution $\mathbf{Q}$.
   (c) Update
      (i) $\mathcal{B} \leftarrow \mathcal{B} \cup \{w | \mathbf{d_G}(u, w) = Q \text{ and } w \in \mathcal{W}\}$,
      (ii) $\mathcal{R} \leftarrow \mathcal{R} \cup \{w | \mathbf{d_G}(u, w) < Q \text{ and } w \in \mathcal{W}\}$,
      (iii) $\mathcal{W} \leftarrow \mathcal{W} \cap (\mathcal{B} \cup \mathcal{R})^c$.
(3) Output $\mathcal{B}$.

Now, the following randomized algorithm outputs a solution which is an ε-approximation of MWIS in expectation for any graph with constant doubling dimension ρ whenever its graph decomposition subroutine achieves (ε, Δ)-decomposition for some constant $\Delta > 0$ [6]. It runs in $O(n)$ time for any constant ρ and ε.

MWIS Approximation (ε, K)

(1) For the given graph G, use the decomposition algorithm with parameters (ε, K) to obtain decomposition of G.
   (a) Let $\mathcal{B}$ be the output of the decomposition algorithm.
   (b) The $V - \mathcal{B}$ is divided into connected components with vertex sets $\mathcal{R}_1, \ldots, \mathcal{R}_L$ (L is some integer).
   (c) Let $G_1 = (\mathcal{R}_1, E_1), \ldots, G_L = (\mathcal{R}_L, E_L)$ be the corresponding disjoint subgraphs of G.
   (d) Let $\mathcal{I}(G_1), \ldots, \mathcal{I}(G_L)$ be set of independent sets of $G_1, \ldots, G_L$ respectively.
(2) For $\ell = 1, \ldots, L$ find

$$\mathbf{x}^*(G_\ell) \in \arg\max \left\{\mathbf{w}^T \mathbf{x} : \mathbf{x} \in \mathcal{I}(G_\ell)\right\}.$$

   (a) The above computation can be done by dynamic programming in $O(2^{|\mathcal{R}_\ell|})$ operations for graph G_ℓ.
(3) Output $\hat{\mathbf{x}} = \cup_{\ell=1}^L \mathbf{x}^*(G_\ell)$ as the candidate for approximate maximum weight independent set of G.

A deterministic algorithm that always outputs ε-approximation of MWIS can be obtained by derandomization of the **MWIS Approximation** [6].

V. Nodes in a plane bounded in one dimension

In [11], we presented a strong polynomial time algorithm that decides the membership of an arbitrary link demand vector in the feasible region of a wireless network where the hosts are confined to a fixed-width slab. One motivation for considering such a scenario is IVHS applications, where the nodes are constrained to a road of bounded width but infinite length. In this section we extend that to get a polynomial algorithm for the end to end demand vector in the feasible region of this class of wireless networks. This contrasts with the other sections because in this case it is an **exact** algorithm, not an approximation.

The locations of the nodes on the fixed-width slab are identified by the point-set $V \subseteq \{(x, y) \in R^2 \mid 0 \leq y \leq w\}$ for some constant $w > 0$. Two nodes, $p, q \in V$ that are separated by a distance less than or equal to r_C have an edge, $l \in E$. Links l_1, l_2 interfere iff (a) $\alpha(l_1), \alpha(l_2), \beta(l_1), \beta(l_2)$ are not all distinct or (b) $d(\alpha(l_1), \beta(l_2)) \leq r_I$ or $d(\alpha(l_2), \beta(l_1)) \leq r_I$, where $r_I > r_C$ is the interference radius. Consider the *adjoint graph*, $\hat{G} = (Q, \hat{E})$, whose vertex set, Q, corresponds to the wireless links, E. For each $l \in E$, we have a corresponding $q \in Q$ with its location at the midpoint of $\alpha(l), \beta(l)$. $q_1, q_2 \in Q$ have an edge between them iff the corresponding links in E interfere. $\hat{G}$ has the following property [11]: For any $q_1, q_2 \in Q$, $d(q_1, q_2) \leq d_{min} \Rightarrow (q_1, q_2) \in \hat{E}$ and $d(q_1, q_2) > d_{max} \Rightarrow (q_1, q_2) \notin \hat{E}$ where $d_{min} = r_I - r_C$ and $d_{max} = r_I + r_C$.

If $\widehat{Q} \subseteq Q$, let:

$\min\{\widehat{Q}\} := \min\left\{ \left\lfloor \frac{x}{d_{max}} \right\rfloor \mid x \in \widehat{Q} \right\}$ and

$$\mathcal{B}(Q) := \{\widehat{Q} \subseteq Q \mid \widehat{Q} \text{ is an independent set of } \widehat{G},$$
$$\text{and } \forall x \in \widehat{Q}, \left\lfloor \frac{x}{d_{max}} \right\rfloor = \min\{\widehat{Q}\}\}.$$

Then, the *auxiliary graph* is defined as a directed graph with a vertex set $\{s, t\} \cup \mathcal{B}(Q)$, with edge set, $\widehat{A}(Q)$ given by:

$$\begin{aligned}\{(s, \widehat{Q}) \mid \forall \widehat{Q} \in \mathcal{B}(Q)\} \cup \{(\widehat{Q}, t) \mid \forall \widehat{Q} \in \mathcal{B}(Q)\} \cup \\ \{(s, t)\} \cup \{(\widehat{Q}, \widetilde{Q}) \in \mathcal{B}(Q) \times \mathcal{B}(Q) \mid \min\{\widehat{Q}\} < \min\{\widetilde{Q}\}, \\ \text{and } \widehat{Q} \cup \widetilde{Q} \text{ is an independent set of } \widehat{G}\}.\end{aligned} \tag{15}$$

Paralleling a result from Matsui [8], stated originally in the context of unit-disk graphs (and extended to (d_{min}, d_{max}) graphs in [11]), the cardinality of the set $\mathcal{B}(Q)$ is polynomial in terms of $n(= |V|)$ when elements of Q are distributed within a fixed-width slab. A link demand vector $\widehat{\mathbf{r}}$, indexed by elements of Q, is feasible iff the optimum value of the below polynomial LP is at most 1 [11].

$$\begin{gathered} min \ \sum_{e \in \delta^+(s)} x_e \\ \text{subject to:} \\ \sum_{e \in \delta^+(v)} x_e - \sum_{f \in \delta^-(v)} x_f = 0, \forall v \in \mathcal{B}(Q) \\ \sum_{v \in \{\widehat{P} \in \mathcal{B}(Q) | q \in \widehat{P}\}} \sum_{e \in \delta^+(v)} x_e \geq \widehat{\mathbf{r}}_q, \forall q \in Q \\ \mathbf{x} \geq \mathbf{0}, \end{gathered} \tag{16}$$

where $\delta^+(v)$ ($\delta^-(v)$) denotes the set of edges with vertex v as its origin (terminus) in the auxiliary graph.

The link demand variable $\widehat{\mathbf{r}}_q$ in equation (16) can now be decomposed into components that satisfy each of the m S-D pairs as $\widehat{\mathbf{r}}_q = \sum_{j=1}^m f_q^j$. The constraints of equation (16) are then augmented with appropriate flow requirements to ensure the link demand vectors, f_q^j support the end-to-end demand vectors for each S-D pair to get the overall LP stated below.

Theorem 13: For a given end-to-end rate vector $\mathbf{r} \in \mathbb{R}_+^m$, we get the following polynomial LP:

$$\begin{gathered} min \ \sum_{e \in \delta^+(s)} x_e \\ \text{subject to:} \\ \sum_{e \in \delta^+(v)} x_e - \sum_{f \in \delta^-(v)} x_f = 0, \forall v \in \mathcal{B}(Q) \\ \sum_{v \in \{\widehat{P} \in \mathcal{B}(Q) | q \in \widehat{P}\}} \sum_{e \in \delta^+(v)} x_e \geq \sum_{i=1}^m f_q^i, \forall q \in Q \\ \sum_{(\mathbf{s}_j, \mathbf{p}) \in E} f_{(\mathbf{s}_j, \mathbf{p})}^j - \sum_{(\mathbf{p}, \mathbf{s}_j) \in E} f_{(\mathbf{p}, \mathbf{s}_j)}^j = \mathbf{r}_j, \forall \text{ sources } \mathbf{s}_j \\ \sum_{(\mathbf{p}, \mathbf{d}_j) \in E} f_{(\mathbf{p}, \mathbf{d}_j)}^j - \sum_{(\mathbf{d}_j, \mathbf{p}) \in E} f_{(\mathbf{d}_j, \mathbf{p})}^j = \mathbf{r}_j, \forall \text{ sinks } \mathbf{d}_j \\ \sum_{(\mathbf{p}, \mathbf{v}) \in E} f_{(\mathbf{p}, \mathbf{v})}^j = \sum_{(\mathbf{v}, \mathbf{p}) \in E} f_{(\mathbf{v}, \mathbf{p})}^j \forall v \notin \{s_j, d_j\}, j \in [m] \\ f_q^j \geq \mathbf{0} \quad \forall q \in Q, j \in [m] \\ \mathbf{x}_e \geq 0 \quad \forall e \in \widehat{A}(Q) \end{gathered} \tag{17}$$

The end-to-end rate vector $\mathbf{r}$ is feasible iff the optimum value above is at most 1.

VI. Conclusion

Since determining the feasibility of a rate vector is a fundamental question in Cross layer design and optimization, we believe that the insights derived from our work on the n^2-dimensional unicast capacity could have impact on the design of wireless mesh networks in the future.

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