Optimal Control of a Broadcasting Server

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***Abstract*— A stochastic control problem motivated by broadcast applications is considered in this paper. A natural queueing model abstraction in which each service to a queue clears all the customers *at once* is adopted, which can also be considered as a *batch processing* queueing model with infinite batch size. Each broadcast can be charged a non-negative cost. In addition, there is a cost whose rate is given as a function of the number of customers waiting in the system at any point. For any cost rate which is a convex function in the number of customers, it is shown that the optimal control is of the threshold type in order to minimize the infinite horizon discounted cost. This result complements the existing literature on batch processing queueing models that have typically only considered monotone costs. For a system with two classes of customers where each service can clear all customers of any given class, with monotone waiting costs and zero service costs, we show that the optimal control can be represented as a double-switch curve in the two dimensional state space. The structure of the optimal policy for multiple queues is a natural next question, and an interesting future direction is to explore the performance of simple *index policies*.**

I. The Model

Consider a system with a dynamic audience interested in a common broadcast from a central server. This can be modeled as a queuing system in continuous time with customers (the audience) arriving according to a Poisson process of intensity λ. The server has the ability to service the audience with broadcasts separated by an exponentially distributed random duration, whose maximum rate is μ (and can be controlled to any value between 0 and μ). Whenever a broadcast is made, the entire audience present in the system at that instance is served (i.e., the total number of customers is reduced to 0 at that instance). There are non-negative costs associated with each broadcast, and also for holding customers in the system, which is specified by a cost rate function that depends on the number of customers in the system at any given time. This cost rate function could be for instance, linear, or more generally even convex in the number of customers waiting. Our aim is to minimize the infinite horizon discounted cost, and to understand the structure of the associated optimal policies.

A more general problem is to consider multiple classes of customers (each belonging to a separate queue) where the server has to also decide which class to serve for each broadcast. In this case, even if there is no cost associated with each service, and the cost rate as a function of the customer state is monotone, the server has to be operated at the maximum rate μ, as in the single queue case. However, we are still left with a scheduling problem of deciding which queue to serve in this situation. A few motivating scenarios for the general broadcast queueing model (i.e., each service clears the entire queue) and the cost models we consider are appropriate before we proceed further:

Broadcast Scheduling: The general broadcast scheduling problem arises in applications where a central server has multiple pages with customer requests for each page arriving independently. Each service can satisfy all outstanding requests for any single page. The aim is usually to minimize either the average waiting time for the page requests, or to minimize the maximum waiting time. A large body of work in the database and algorithms literature has focused on scheduling for the broadcasting model (eg: [16], [5], [12], [4], [11], [7]). A strong emphasis is on competitive analysis, in which oblivious online policies and optimal offline algorithms that have a precise knowledge of the future sample path of the customer arrivals are analyzed. To put the stochastic control model in perspective with this alternate line of investigation, one could view this as a middle ground between the omniscient offline algorithm and the completely oblivious online algorithm, since the assumption of Poisson arrivals amounts to a partial knowledge on the sample paths of customer arrivals. In this context, [19] studies the batch processing problem in multiple queues. A variant with constant service time was studied also in [14]. In this paper, we also consider scheduling a broadcast server with two queues with monotone costs in section IV and conclude that a switching structure is optimal.

Wireless Personal Area Networks: Consider wireless hosts in close proximity of each other with high individual link rates. Interference constraints imply that only one link can be active at a time. Because of the high data rate for each individual links, essentially all outstanding packets are cleared whenever a link is activated, and the scheduling problem needs to resolve which link to activate at any given time. Among the myriad applications that wireless networks have found, one challenging and emerging scenario is the wireless personal area network (WPAN) ([1]), which closely fits the wireless model described. A typical WPAN consists of hosts distributed in a very short range that communicate via mutually interfering wireless links, because of which scheduling them optimally becomes a challenging issue.

Motivation for Non Monotone Cost: The reasons motivated in [10] could also be argued for the current model. A non monotone cost model could have utility in flow control problems and also as a technique to understand a multiple queue system of monotone costs by considering a related

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single queue model with non monotone costs. Further, a non monotone cost is also relevant to a situation where the server has a strategic interest in *not* keeping the audience interested in its broadcast as low as possible at all times. Such a possibility is easy to imagine, for instance, in a peer to peer service model with selfish peers, where having too few interested peers on average incurs a high cost for the server because the server is itself predominantly only served by other selfish peers that are actively interested in its own service. On the other hand, keeping too many peers waiting also incurs a progressively higher cost beyond a certain point, because the server then risks being classified as a free loader by its peers, leading to punishment from its frustrated peers in the form of degraded service to itself.

Queueing systems have been studied under the closely related "batch processing" model. In the batch processing model corresponding to a batch size of B, each service to the queue can clear a maximum of B customers. For broadcast, $B = \infty$. Serfozo and Deb, in [9] considered the stochastic control problem of a single queue under batch processing, and proved that the optimal control is of threshold type when the instantaneous cost function is monotone in the number of customers. There is extensive literature on the batch processing model for a single queue ([2], [3], [8], [17], [18]), however, a common theme in all previous work is that the cost rate is generally monotone in the number of customers. On a related note, [10] proved that a threshold control is optimal for the standard queueing service model (i.e. each service clears one customer) with convex cost rate in the number of customers, and without any service cost. While portions of our proof borrow heavily on techniques from [10], the overall argument is based on a novel use of the standard technique of policy iteration.

II. Notation and Preliminaries

We will first consider a single queue. Assume that arrivals to the queue are defined by a Poisson process of rate λ. Whenever the queue is served, all customers in the queue exit at once. Without loss of generality, assume that $\lambda + \mu = 1$. Let $c : Z_+ \mapsto R_+$ denote a non-negative cost rate defined as a function of the number of customers in the queue, which we denote as x_t at time t. We assume that c is convex and also has an appropriate growth restriction to ensure that the infinite horizon discounted cost is well defined. Let c_{sw} denote the constant service cost associated with each broadcast. Assume that for a given time a control value, $0 \le w \le 1$, specifies the (exponential) rate of the broadcast as being $w\mu$. A useful way of interpreting this situation is to look at the rate μ process consistently at all times, and then actually utilizing this *potential* broadcast opportunity with probability w when the control being applied is w, an equivalence which follows directly from the Poisson splitting property.

Now consider a rate 1 Poisson process obtained by adding the arrival process of rate λ and the potential departure process of rate μ. Let τ_n be the n^{th} transition of this net process and let x_n denote x_{τ_n} for simplicity. The discrete time jump Markov process obtained by sampling the system between these transitions has the following transition probabilities defined on Z_+ (this discrete time process is independent of the inter-event times): $p(y/x) = \lambda I\{y = x+1\} + \mu(w(x)I\{y = 0\} + (1 - w(x))I\{y = x\}$, where w denotes the stationary, feedback control as a function of the current state. The infinite horizon discounted cost to be minimized is:

$$E_x^w \int_0^\infty e^{-\alpha t} c(x_t)\, dt + \sum_{k=1}^{\infty} e^{-\alpha \tau_k} \chi\{x_{k-1} \neq 0 \text{ and } x_k = 0\} c_{sw}$$

where E_x^w denotes the expectation with the control w starting at $x_0 = x$, α is the discount factor and χ denotes the indicator function for the conditions given in its argument, and c_{sw} is the non negative service cost associated with each broadcast. The above expression can be shown to be a constant factor of the equivalent cost on the discrete time process by invoking the independence between the inter-event times and the jump process dynamics ([13]). This gives us an equivalent optimization on the discrete time Markov decision process defined above with the following objective function:

$$E_x^w \sum_{k=0}^{\infty} \beta^k \left(c(x_k) + c_s \chi\{x_{k-1} \neq 0, x_k = 0\} \right)$$

where $0 < \beta < 1$ is the discount factor and c_s (rather than c_{sw}) is the service cost for the equivalent discrete time problem. We shall also use a convention that $x_{-1} = 0$. Let w be a $[0, 1]$ valued function on Z_+ denoting the control (i.e., $w(x)\mu$ is the rate of the broadcast server with x customers in the queue). Let U^w denote the value function corresponding to the control w:

$$U^w(x) = E_x^w \sum_{k=0}^{\infty} \beta^k \left(c(x_k) + c_s \chi\{x_{k-1} \neq 0, x_k = 0\} \right) \tag{1}$$

By a simple recursion argument, it can be shown that this value function satisfies a fixed point equation corresponding to the dynamic programming operator for U^w, given by

$$\mathcal{T}^w f(x) = c(x) + \beta(\lambda f(x+1) + \mu(w(x)(f(0) + c_s) + (1 - w(x))f(x)))$$

In other words,

$$U^w = \mathcal{T}^w U^w \tag{2}$$

Further, uniqueness of a solution to the above equation is implied by fixed point theorem for contractions in complete metric spaces, provided we assume the appropriate growth restrictions on $c(x)$ as in [10], [13] (this is not restrictive, unless we need to model a situation with super exponential costs). Let $V(x)$ be the optimal value function, defined as the infimum over arbitrary control policies u, of the expected infinite horizon discounted cost starting at x :

$$V(x)=\inf_u E_x^u \sum_{k=0}^{\infty} \beta^k (c(x_k)+c_s\chi\{x_{k-1}\neq 0, x_k=0\}) \tag{3}$$

The fixed point equation operator for V is:

$$\mathcal{T}f(x)=c(x)+\beta\{\lambda f(x+1)+\mu\min(f(x),f(0)+c_s)\} \tag{4}$$

V is the unique solution to:

$$V=\mathcal{T}V \tag{5}$$

Given the optimal V, an optimal control w would then be:

$$w(x)=\begin{cases}1 & \text{if } V(x)>V(0)+c_s\\ 0 & \text{if } V(x)\leq V(0)+c_s\end{cases}$$

A key result we prove in this paper is the following:

Theorem 2.1: The optimal control is given by the stationary state feedback control $w(x)=I\{x\geq l^*\}$ for a critical threshold l^*. Further,

$$l^*=\min\{l:U_l(l)>U_l(0)+c_s\}$$

III. Proof of Threshold Optimality

To denote w of the form $w(x)=I\{x\geq l\}$, we will from now on write it as w_l and the value function corresponding to it as U_l. It suffices to show that equation (5) is satisfied for the operator T^w corresponding to w defined by a threshold control in order to show that it is an optimal control. First, we recall the definition of a quasiconvex (unimin) function:

Definition 3.1: A function f on Z_+ is quasiconvex (unimin) if $f(x+1)-f(x)\geq 0$ for all $x>y$ whenever $f(y+1)-f(y)>0$

A key element of the argument is to show that:

Theorem 3.2: If $U_l(l-1)\leq U_l(0)+c_s<U_l(l)$, then U_l is quasiconvex.

Proof: Implied by Lemmas 3.3, 3.5 and the hypothesis that $U_l(l-1)<U_l(l)$ ■

Lemma 3.3: If $U_l(l-1)\leq U_l(l)$, then $U_l(x)$ is quasiconvex on $0\leq x\leq l-1$.

Proof: Within this Lemma, we will drop the subscript l and have the convention that U means U_l, and denote $\frac{\beta\lambda}{1-\beta\mu}=\gamma$ (note that $0<\gamma<1$). Further, we denote $c'(x)=c(x)/(1-\beta\mu)$. For $0\leq x\leq l-1$, $U(x)=c(x)+\beta\lambda U(x+1)+\beta\mu U(x)$, which in turn implies:

$$\begin{aligned}U(x)&=c'(x)+\gamma U(x+1)\\ &=c'(x)+\gamma\{c'(x+1)+\gamma U(x+2)\}\\ &\dots\\ &=c'(x)+\gamma c'(x+1)+\gamma^2 c'(x+2)+\dots\\ &\quad \dots+\gamma^{l-x-1}c'(l-1)+\gamma^{l-x}U(l)\end{aligned}$$

Let $\delta'(x)\triangleq c'(x+1)-c'(x)$. δ' is increasing since c' is convex. Also define

$$\Delta U(x)\triangleq U(x+1)-U(x)$$

Using the above definition, and from the fact that

$$U(l-1)=c'(l-1)+\gamma U(l)$$

, one can verify the following relation for $0\leq x\leq l-2$:

$$\begin{aligned}\Delta U(x)=&\delta'(x)+\gamma\delta'(x+1)+\dots\\ &\dots+\gamma^{l-x-2}\delta'(l-2)+\gamma^{l-x-1}(U(l)-U(l-1))\end{aligned}$$

A sufficient condition for $U(x)$ to be quasiconvex on $0\leq x\leq l-1$ is the existence of a $\xi(x)>0$ such that $\frac{\Delta U(x)}{\xi(x)}$ is increasing[1] for $0\leq x\leq l-2$. We will now show this for the choice of $\xi(x)=1-\gamma^{l-x-1}>0$ for $0\leq x\leq l-2$. Since $\frac{\Delta U(x)}{\xi(x)}$ depends on the function δ' in a linear fashion, we just need to verify that $\frac{\Delta U(x)}{\xi(x)}$ is increasing when δ' is a constant, and when it is of the form $I\{x\geq b\}$. First if $\delta'(x)=a$ for any constant a, we have:

$$\begin{aligned}&\frac{\Delta U(x)}{\xi(x)}\\ &=a\frac{1+\gamma+\dots+\gamma^{l-x-2}}{1-\gamma^{l-x-1}}+\frac{\gamma^{l-x-1}}{1-\gamma^{l-x-1}}(U(l)-U(l-1))\\ &=\frac{a}{1-\gamma}+\frac{\gamma^{l-1}}{\gamma^x-\gamma^{l-1}}(U(l)-U(l-1))\end{aligned}$$

, which is increasing in x since (1) $\gamma<1$ and (2) $U(l)>U(l-1)$, by the hypothesis of the Lemma. Next let $\delta'(x)=I\{x\geq b\}$. We then have (with the convention that if $b>l-2$, the appropriate terms below will be 0, and hence increasing by default):

$$\begin{aligned}&\frac{\Delta U(x)}{\xi(x)}=\\ &\frac{\gamma^{b-x}+\dots+\gamma^{l-x-2}}{1-\gamma^{l-x-1}}+\frac{\gamma^{l-x-1}}{1-\gamma^{l-x-1}}(U(l)-U(l-1))\\ &=\frac{\gamma^b+\dots+\gamma^{l-2}}{\gamma^x-\gamma^{l-1}}+\frac{\gamma^{l-1}}{\gamma^x-\gamma^{l-1}}(U(l)-U(l-1))\end{aligned}$$

, which is again increasing with x by the hypothesis of the Lemma. ■

Lemma 3.4: $U_l(x)$ is quasiconvex for $x\geq l$.

Proof: This can be proved by considering a coupled process and using an argument similar to [10]. The complete proof is given in the appendix. ■

Lemma 3.5: If $U_l(l-1)\leq U_l(0)+c_s<U_l(l)$, then $U_l(x)$ is increasing on $x\geq l$

Proof: We already know that U_l is quasiconvex for $l\leq x$ from Lemma 3.4. Hence, it is sufficient to show that $U_l(l+1)>U_l(l)$ to prove that it is increasing on $x\geq$

[1]Throughout this paper, 'increasing' and 'decreasing' mean 'non-decreasing' and 'non-increasing' respectively.

l. For the rest of the proof in this Lemma, we will again implicitly drop the subscript l in U_l. Assume to the contrary that $U(l+1) \leq U(l)$. Then:

$$\begin{aligned}
&U(l) \\
&= c(l) + \beta\{\lambda U(l+1) + \mu(U(0) + c_s)\} \\
&\leq c(l) + \beta U(l) \, (\because U(0) + c_s < U(l), U(l+1) \leq U(l))
\end{aligned}$$

and,

$$\begin{aligned}
&U(l-1) \\
&= c(l-1) + \beta\{\lambda U(l) + \mu U(l-1)\} \\
&\geq c(l-1) + \beta U(l-1) \quad (\because \; U(l) > U(l-1))
\end{aligned}$$

The above two inequalities imply:

$$\begin{aligned}
&\Delta U(l-1) \leq \delta(l-1) + \beta \Delta U(l-1) \\
&\Rightarrow 0 < (1-\beta)\Delta U(l-1) \leq \delta(l-1)
\end{aligned}$$

Since δ is increasing, this also means that $\delta(l) > 0$. Then, again by a certain coupling argument similar to Lemma 3.4, we have $U(l+1) > U(l)$. More precisely, consider the coupled process of the proof of Lemma 3.4. For $x_0 = l$ and τ, the stopping time as defined in proof of Lemma 3.4, x_k is an increasing sequence for $0 \leq k \leq \tau - 1$. Hence, $\delta(x_k) \geq \delta(l) > 0$ for $0 \leq k \leq \tau - 1$. Thus, using equation (13), $\Delta U(l) = E_{x_0=l} \sum_{k=0}^{\tau-1} \beta^k \delta(x_k) > 0$, which in turn implies that $U_l(x)$ is increasing on $x \geq l$. ■

Lemma 3.6: If $c(x)$ is convex, unless it is decreasing on all x, we have for some l large enough: $U_l(l) > U_l(0) + c_s$.

Proof: Suppose $U_i(i) \leq U_i(0) + c_s$ for all $i < l$. For any $i < l$, consider a Markov decision problem where the only decision variable is at state $x = i$, with the rest of the control fixed to match the threshold $i+1$ control, w_{i+1}. Now apply Policy iteration to the threshold i policy on the above MDP. Since $U_i(i) \leq U_i(0) + c_s$, an optimal control is to set $w(i) = 0$ (i.e. don't serve at $x = i$), and policy iteration results in w_{i+1}, the threshold-$i+1$ control. Hence, U_{i+1} is componentwise less than U_i. Specifically, this means $U_i(0)$ is a decreasing sequence for $i \leq l$, implying that $U_l(0) \leq U_1(0)$. Since c is not decreasing and is convex, for some l large enough, we have $c(l) > U_1(0) + c_s$. For such an l, if we also have $U_i(i) \leq U_i(0) + c_s$ for all $i < l$, then $U_l(l) = c(l) + \beta\{\lambda U_l(l+1) + \mu(U_l(0) + c_s)\} > c(l) \geq U_1(0) + c_s \geq U_l(0) + c_s$. ■

Theorem 3.7: The optimal control is a threshold policy corresponding to the threshold l^* given by

$$l^* = \min\{l : U_l(l) > U_l(0) + c_s\} \tag{6}$$

Proof: Suppose l^* is infinite. Then, the contrapositive of Lemma 3.6 implies that $c(x)$ is decreasing, in which case it is clear that an optimal policy is to never serve, which corresponds to a threshold $l^* = \infty$ optimal control. Thus, we now only need to argue about the case where l^* is finite. Now suppose

$$U_{l^*}(l^*-1) > U_{l^*}(0) + c_s \tag{7}$$

Then consider the Markov decision subproblem where the the control is fixed to match the threshold l^* control for all x except for $x = l^* - 1$, which is the only decision variable. Then, by an application of policy iteration for this subproblem, we conclude that U_{l^*-1} strictly improves U_{l^*}. Now if the following is true:

$$U_{l^*-1}(l^*-1) \leq U_{l^*-1}(0) + c_s \tag{8}$$

, then we can again consider the Markov decision subproblem where the only decision variable is at $x = l^* - 1$ and everything else is fixed to match the threshold l^* control. Such consideration implies that U_{l^*} improves U_{l^*-1}, a contradiction to what we just concluded above. Hence equation (8) must be false and

$$U_{l^*-1}(l^*-1) > U_{l^*-1}(0) + c_s$$

which contradicts the definition of l^* in equation (6). Hence, the assumption in equation (7) is false and we conclude that:

$$U_{l^*}(l^*-1) \leq U_{l^*}(0) + c_s$$

This means that l^* satisfies the hypothesis for Theorem 3.2 and hence is quasiconvex. This implies:

$$U_{l^*}(x) \begin{cases} \leq U_{l^*}(0) + c_s & \text{if } x \leq l^* - 1 \\ > U_{l^*}(0) + c_s & \text{if } x \geq l^* \end{cases}$$

Therefore, it also satisfies the fixed point equation corresponding to the optimal value function dynamic programming operator given in equation (4). ■

IV. Scheduling Two Queues

In this section we shall consider the case of two queues. Unlike the single queue case for which we were able to tackle the convex cost model (and the monotone cost model was handled in [9]), we will only consider a monotone cost in the number of waiting customers for two queues. More explicitly, there are two classes of customers who arrive according to independent Poisson processes of rates λ_1 and λ_2. We also have a broadcasting server of rate μ. Assume without loss of generality that $\lambda_1 + \lambda_2 + \mu = 1$. The is no service cost. Let the cost rate be given as $c(x_1, x_2)$ when the number of customers in queues 1 and 2 is x_1, x_2 respectively. We shall assume that c is non decreasing in (x_1, x_2). Again, although a continuous time system, the time integrals of the instantaneous cost (both discounted as well as long run average) can be conveniently cast in terms of the discrete time jump processes because of the independence of inter-event times with respect to the states (which comes from the Poisson arrivals and service processes). Let β be the equivalent discount factor for this discrete time problem. Let c_1 and c_2 be the equivalent service costs for queue 1 and 2 respectively for the equivalent discrete time problem. The n step cost function starting at state $(x_1(0), x_2(0))$ is (where

u denotes control and the evolution of the state is implicitly as per control u):
$V_n(x_1,x_2) = \inf_u E^u_{(x_1(0),x_2(0))} \sum_{k=0}^{n-1} \beta^k c(x_1(k), x_2(k))$
Via dynamic programming, we can recursively characterize V_n as: $V_0 \equiv 0$, and:

$$\begin{aligned} V_{n+1}(x_1,x_2) = c(x_1,x_2) + \\ \beta\{\lambda_1 V_n(x_1+1,x_2) + \lambda_2 V_n(x_1,x_2+1) + \\ \mu \min(V_n(x_1,0), V_n(0,x_2))\} \end{aligned} \tag{9}$$

The optimal control action with n steps to go at state (x_1,x_2) is given by:

$$u_n(x_1,x_2) = \begin{cases} 2 & \text{, if } V_n(x_1,0) \le V_n(0,x_2) \\ 1 & \text{, otherwise.} \end{cases} \tag{10}$$

In the above description, the control variable $u_n(x_1,x_2)$ denotes the queue to be served at state (x_1,x_2).

Remark 1: By letting $n \to \infty$ it can be argued that V_∞ exists and V_n converges to it, and V_∞ also inherits the properties of V_n that are shown below via induction, including the switching structure, because the set of functions satisfying them is closed under point-wise limits.

Let $(x_1,x_2) \prec (y_1,y_2) \Leftrightarrow x_1 \le x_2$ and $y_1 \le y_2$

Lemma 4.1: For any $x \in Z^2_+, y \in Z^2_+$ such that $x \prec y$, $V_n(x) \le V_n(y)$.

Proof: Let $x = (x_1,x_2), y = (y_1,y_2)$ be such that $x \prec y$. The assertion holds for $n = 0$ from the monotonicity of c. We also have:

$$\begin{aligned} &V_{n+1}(y) - V_{n+1}(x) = c(y_1,y_2) - c(x_1,x_2) \\ &+ \beta\lambda_1(V_n(y_1+1,y_2) - V_n(x_1+1,x_2)) \\ &+ \beta\lambda_2(V_n(y_1,y_2+1) - V_n(x_1,x_2+1)) \\ &+ \beta\mu(\min(V_1(y_1,0),V_n(0,y_2)) - \min(V_n(x_1,0),V_n(0,x_2)))\} \end{aligned}$$

If $x \prec y$, we also have $(x_1+1,x_2) \prec (y_1+1,y_2), (x_1,0) \prec (y_1,0)$, etc. If the assertion holds for n, one can easily check that that this implies that each of the above terms is non-negative. Therefore, it also holds for $n+1$. ∎

Theorem 4.2: The optimal control with n steps to go is given by a switch curve:

$$u_n(x_1,x_2) = \begin{cases} 2 & \text{, if } x_2 \ge s_n(x_1) \\ 1 & \text{, otherwise.} \end{cases} \tag{11}$$

where

$$s_n(x) = \min\{y : V_n(x,0) \le V_n(0,y)\} \tag{12}$$

Proof: Follows from interpreting equation (10) using Lemma 4.1. ∎

V. Conclusion and Further Work

The broadcast service queueing model which corresponds to batch processing with a batch size infinity was considered. For a single queue, we have seen that a threshold control is optimal not only for monotone, but also for convex cost function and with constant service costs. For two queues with a monotone cost rate, we see that the optimal control

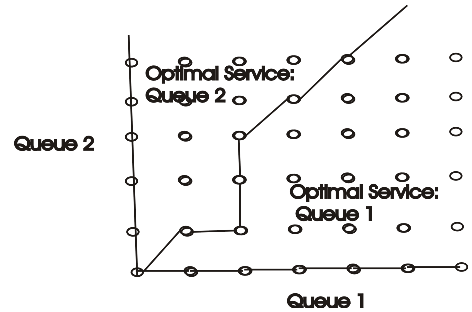

Fig. 1. An Illustration of the State space for 2 queues and the optimal control that is proved in theorem 4.2

can be described by a switch curve. A natural question is whether the optimal policy has a simple structure for $n > 2$ queues. For simple algorithms, one might want to restrict attention to *index rules* for scheduling. By an index rule, we mean a policy which can be computed by comparing scores for each queue independently. Formally, at a state $(x_1,\ldots,x_n)$, we would like to be able to describe the policy as $u(x_1,\ldots,x_n) = \arg\max_{i\in[n]}\{\psi_i(x_i)\}$ where ψ_i's are some functions describing the policy [2]. Yet another useful line of pursuit might be to obtain suboptimal, yet provably useful index rules for asymptotically large number of queues. For instance, the popular longest wait first algorithm (LWF) ([6], [5]) in the deterministic setting is approximately equivalent to using an index policy corresponding to scores of $\frac{x_i}{\sqrt{\lambda_i}}$ in the stochastic setting after some simple calculations.

VI. Acknowledgements

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VII. Appendix

Proof: [Complete Proof of Lemma 3.4] Given an integer l, define a Markov process on Z^2_+ with the following transitions:

$$\begin{aligned} p((x',y')|(x,y)) = \lambda I\{x' = x+1, y' = y+1\} + \\ \mu I\{x' = x(1 - I\{x \ge l\}), y' = y(1 - I\{y \ge l\})\} \end{aligned}$$

Then, corresponding to any initial state (x,y) the above coupled process has marginals which are identical to the individual processes. Although the condition (3) of [10] doesn't hold anymore, restrict attention to any $x \ge l$ and consider the process started in $(x, x+1)$. Then, $y_k - x_k$ takes values in $0,1$ and is decreasing in k. Let E_x denote the expectation under this starting condition. Let

$$\tau = \min\{k \ge 0 : x_k = y_k\}$$

As in [10], let:

$$r(x) = \frac{U(x+1) - U(x)}{E_x \sum_{k=0}^{\tau-1} \beta^k}$$

[2] It is well known that optimal policies can be described by index rules for multi armed bandit problems ([15])

On $x \geq l$, it can be shown that r is increasing, which implies U is quasiconvex on the same domain. For $x \geq l$:

$$U(x+1) - U(x) = E_x \sum_{k=0}^{\infty} \beta^k (c(y_k) - c(x_k)) = E_x \sum_{k=0}^{\tau-1} \beta^k \delta(x_k) \tag{13}$$

so that:

$$r(x) = \frac{E_x \sum_{k=0}^{\tau-1} \beta^k \delta(x_k)}{E_x \sum_{k=0}^{\tau-1} \beta^k}$$

Since δ is increasing and since r depends linearly on δ it suffices to verify that it is increasing for constants and for functions of the form $I\{x \geq b\}$. Since r is a constant if δ is a constant or if $\delta(x) = I\{x \geq b\}$ where $b \leq x$ since $x \geq l$, we only have to check for $I\{x \geq b\}$ where $b > x$. Let

$$\sigma = \tau \wedge \min\{k : x_k = x + 1\}$$

Then $r(x)$ can be compared favorably with $r(x+1)$ by writing (using $b > x$):

$$E_x \sum_{k=0}^{\tau-1} I\{x_k \geq b\} \beta^k = E_x[\beta^\sigma I\{\sigma < \tau\}] E_{x+1} \sum_{k=0}^{\tau-1} I\{x \geq b\} \beta^k$$

and

$$E_x \sum_{k=0}^{\tau-1} \beta^k = E_x \sum_{k=0}^{\sigma-1} \beta^k + E_x[\beta^\sigma I\{\sigma < \tau\}] E_{x+1} \sum_{k=0}^{\tau-1} \beta^k$$

■

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