Relaying a Fountain Code

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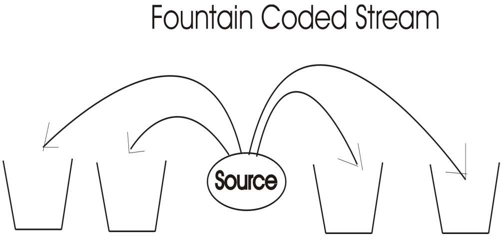
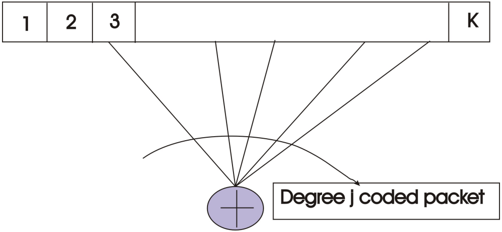
Fountain Codes



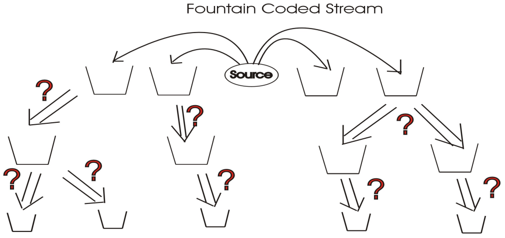
- Each client collects enough packets to decode
- Coding doesn't depend on the erasure probabilities - Rateless!
- Most popular examples: LT, Raptor.

Luby Codes



- Average degree - $O(\log k)$ $\Rightarrow$ Logarithmic Per Symbol Complexity
- $k + O(\sqrt{k} \log^2 k)$ coded packets sufficient $\Rightarrow$ Rate Optimal and very low overhead
- Decoding - Iterative BP decoding

Multiple hops



- Simply Forwarding - lose mincut capacity
- Decode and Reencode - Delay
- Random Linear Codes - $O(k)$ Complexity
- Chunked Codes [Harvey et. al. 2006] - $O(\log^2 k)$ complexity
- Trade off Schemes for Line networks [Pakzad et. al. 2005]
- Can we achieve the optimality of single hop?

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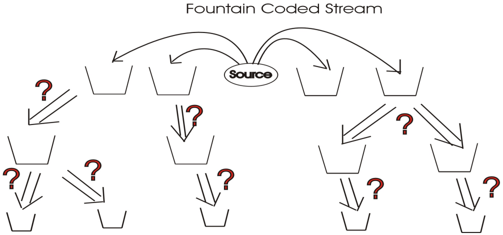

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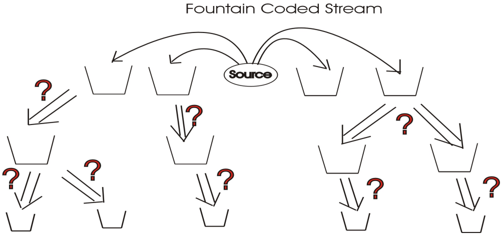

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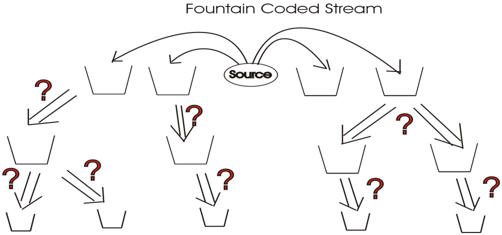

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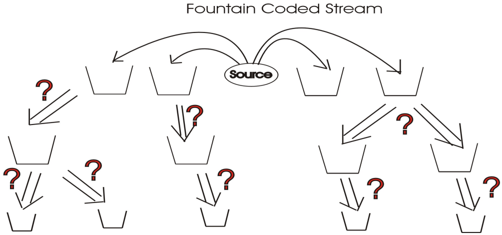

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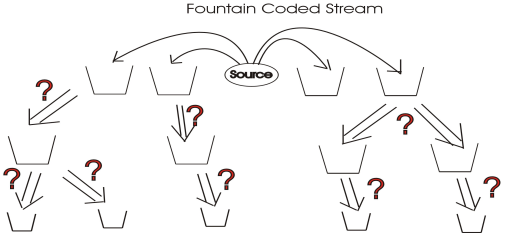

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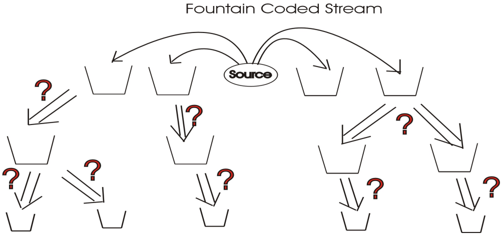

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- No Delay because of Decode and Re-encode
- Rateless
- Throughput rate to any node = Its min cut capacity from the source
- Complexity and Overhead similar to LT codes at all nodes

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Assumptions

- Tree Network
- Discrete Memoryless Erasure Channels
- Universal Upper bound on Erasure probabilities

Challenges

- Online Encoding
  - Intermediate Nodes can only access packets sequentially as received.
- Re-encoding the Coded packets
  - Intermediate nodes should be able to re-encode the coded packets without waiting to decode.

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A toy problem

i^{th} coded packet has to be a combination of the first i packets alone

- Generate a random set of $k + o(k)$ *symbols* according to the LT encoding process.
- Run a mock decoder. Let π be the sequence in which we see decoded packets.
- Decodability $\Rightarrow$ the coded packet used in the i^{th} step of decoding was a combination involving only the first i decoded packets.
- Do actual encoding *online* by assigning packet indices in the sequence defined by π!

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Re-encoding coded packets

- Do concatenated coding - slap on successive layers of the same code at each hop
- Fix a sequence of block lengths, $k = k_0 \ldots k_n$ along the hops.
  - subject to: k_i coded packets at node i is enough to recover the k_{i-1} packets that were recoded by node $i-1$ w.h.p
- $k_i \leq k_0(1 + \frac{\log^2 \frac{kn}{\delta}}{\sqrt{k}})^i$
- Overhead doesn't accumulate over hops, since $k_n = O(k_0)$ if we set $k_0 = \Omega(n^3)$.
- Complexity of encoding $\sim$ LT coding on a block length k_i
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Terminology

- **Definition**

  An **Online Code Copy** is an ordered sequence of k_{i+1} **code symbols** that can be generated in an "*online fashion*".

- **Definition**

  A **Code Matrix** is a $T(k) \times k_{i+1}$ random matrix of *Code Symbols* in which each row is an independent *Online Code Copy*.

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  The **Online phase** at node i is defined to be the period until the time slot at which node i collects a total of k_{i+1} coded packets.

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  State is the number of packets successfully collected so far.

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CodeMatrix

CODE BLOCK at node *i*

Code copy

c_{11}	c_{12}	c_{13}	
c_{21}	c_{22}		
c_{31}			
c_{T1}			

Dimensions: T X k_{i+1}

Number of indices used to generate the symbols: k_i

Algorithm

Procedure $LT - RELAY$ (node i)

1. First generate:
   (i) A random *code matrix*, $\mathcal{M}_i = [c_{ij}]_{1 \leq i \leq T, 1 \leq j \leq k_{i+1}}$
   (ii) An independent random *online code copy*, $\mathcal{R}_i = \{\theta_j\}_{1 \leq j \leq k_{i+1}}$
2. *Online phase (i.e. while in state $j, 0 \leq j \leq k_{i+1}$):*
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   (ii) Remaining Slots: Pick $\hat{c}$ uniformly at random from the j^{th} column of $\mathcal{M}_i$. If not previously picked, send a packet coded according to $\hat{c}$. Else, it becomes an *idle slot*.
3. Beyond the online phase, generate independent coded packets at each time slot using the standard LT coding procedure.

- **Theorem**

  (w.h.p.)N_i, Number of idle slots at node i satisfies $N_i \leq \log k$

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Uniformity in Code Symbol Selection

Theorem

Let $\chi = \{0,1\}^{T(k)\times k}$ *denote the ensemble of all possible realizations of the random matrix,* Λ*. For* $\Psi = [\psi_{ij}] \in \chi$ *and for any* $S \subset \{1,\ldots,T(k)\} \times \{1,\ldots,k\}$*, denote*

$$W_S(\Psi) = \sum_{(i,j)\in S} \psi_{ij}$$

Take any $E \subset \{1,\ldots,T(k)\} \times \{1,\ldots,k\}$*, with* $|E| = r$*, a constant represented as* $E = \{e_1,\ldots,e_r\}$*. For any* $\phi = (\phi_1,\ldots,\phi_r) \in \{0,1\}^r$ *let* $\Theta_\phi = \{\Psi \in \chi : \psi_{e_j} = \phi_j$ *for* $1 \leq j \leq r\}$*. Then, as* $k \to \infty$*,* $P(\Theta_\phi)$ *depends solely on* $\sum_{i=0}^{r} \phi_i = W_E(\Psi)\ \forall\ \Psi \in \Theta_\phi$ *.*

Min Cut Capacity

- **Theorem**

  Given that (i) the subset of code symbols from $\mathcal{M}_i$ used is uniformly random and (ii) t is past the online phase, the set of all coded packets generated till time slot t forms an LT code.

- **Proof.**

  Packets generated were the union of

  1. $\mathcal{R}_i$
  2. An (almost) uniform random subset of code symbols from $\mathcal{M}_i$
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  (w.h.p.) Assuming monotonically increasing erasure probabilities, the **first** k_i *packets collected at node* i *can be decoded to recover the* k_{i-1} *packets that were recoded by node* $i-1$.

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  The code described is capacity achieving. That is, packets are transmitted from the source to the node i *at a rate equal to* $\min_{1\leq j\leq i}(1-\epsilon_j)$.

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Conclusion

Have shown:

- Min Cut Capacity to every node.
- Ratelessness
- Order optimal delay
- Low overhead
- Low complexity.
- On arbitrarily large tree networks!

Thanks

Thank you!

For more details, please
(a) Talk to me, or
(b) Read a preprint from
http://decision.csl.uiuc.edu/~gummadi2/papers/fountain.pdf