Optimal Control of a Broadcasting Server

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Outline of the Talk

- **Introduction**
  - The Basic Broadcast Server Queueing Model
  - Motivation for studying the Model

- **Single Queue Problem**
  - Objective and the convex cost model with broadcast costs
  - Main Result: Threshold property of the optimal control
  - Proof Outline

- **Two Queues**
  - Cost Model
  - Switch Curve Property of the Optimal Control

- **Conclusion and Further Work**
The Basic Broadcast Queueing Model

- Continuous time system
- Poisson Arrivals of rate $\lambda$
- Exponential Server of rate $\mu$
- Each service clears the entire queue
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Motivation for studying this model

Broadcast Scheduling

- \( \Lambda_1 \)
- \( \Lambda_2 \)
- \( \ldots \)
- \( \Lambda_n \)

- Each link rate without interference is very high
- Schedule them so as to minimize cost given as some function of the queue sizes
Motivation for studying this model

§ Broadcast Scheduling

- $\Lambda_1$
- $\Lambda_2$
- $\ldots$
- $\Lambda_n$

§ Batch processing systems with large batch size
Motivation for studying this model

- Broadcast Scheduling
  - $\mathbf{A}_1$  
  - $\mathbf{A}_2$  
  - $\mathbf{A}_n$

- Batch processing systems with large batch size

- High Interference Scheduling - WPAN
  - $n$ mutually interfering links in close proximity
  - Each link rate without interference is very high
  - Schedule them so as to minimize cost given as some function of the queue sizes
Objective: Single Queue

- \( c(x) \) is a cost rate for holding \( x \) customers in the system
- \( c_s \) is an additional cost per broadcast
- At state \( x \) we operate the server at rate \( w(x)\mu \) for \( 0 \leq w(x) \leq 1 \)
- Describe the optimal control \( w(x) \) to minimize:

\[
E^w_x \int_0^\infty e^{-\alpha t} c(x_t) \, dt + \sum_{k=1}^\infty e^{-\alpha \tau_k} \mathbb{I}\{x_{k-1} \neq 0 \text{ and } x_k = 0\} c_s
\]
Cost Models

- Previous work on batch service models shows that \( w(x) \) is threshold type for monotone costs, \( c(x) \).

- Current Work: any convex \( c(x) \).

- **Practical motivation** for convex cost on single queue:
  1. p2p system with strategic cost model abstraction
  2. Heuristics to decompose multiple queue systems to single queue.
Main Result: Single Queue

Theorem

A threshold policy is optimal for discounted infinite horizon cost for convex cost rate $c(x)$ and constant service cost
Main Result: Single Queue

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A threshold policy is optimal for discounted infinite horizon cost for convex cost rate $c(x)$ and constant service cost.

- Minimize

$$E^w_x \int_0^\infty e^{-\alpha t} c(x_t) \, dt + \sum_{k=1}^{\infty} e^{-\alpha \tau_k} \mathbb{1}\{x_{k-1} \neq 0 \text{ and } x_k = 0\} c_s$$
Theorem

A threshold policy is optimal for discounted infinite horizon cost for convex cost rate \( c(x) \) and constant service cost

- Minimize

\[
E_x^w \int_0^\infty e^{-\alpha t} c(x_t) \, dt + \sum_{k=1}^\infty e^{-\alpha \tau_k} \mathbb{I}\{x_{k-1} \neq 0 \text{ and } x_k = 0\} c_s
\]

- Equivalent to a discrete time problem for minimizing:

\[
U^w(x) = E_x^w \sum_{k=0}^\infty \beta^k (c(x_k) + c_s \mathbb{I}\{x_{k-1} \neq 0, x_k = 0\})
\]
Single Queue

- Dynamic programming operator, $\mathcal{T}$ defined as:

$$\mathcal{T} f(x) = c(x) + \beta \left\{ \lambda f(x+1) + \mu \min(f(x), f(0) + c_s) \right\}$$
Single Queue

- Dynamic programming operator, \( \mathcal{T} \) defined as:

\[
\mathcal{T} f(x) = c(x) + \beta \{ \lambda f(x + 1) + \mu \min(f(x), f(0) + c_s) \}
\]

- Optimal Value function:

\[
V(x) = \inf_{E_x^u} \sum_{k=0}^{\infty} \beta^k (c(x_k) + c_s \mathbb{1}_{\{x_{k-1} \neq 0, x_k = 0\}})
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Single Queue

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• From Dynamic Programming argument, $V$ satisfies:

\[
V = \mathcal{T} V
\]
Proof of Threshold Optimality

For a given control \( w(x) \), the value function

\[
U^w(x) = E_x^w \sum_{k=0}^{\infty} c(x_k) \beta^k
\]

satisfies a fixed point eqn for:

\[
T^w f(x) = c(x) + \beta(\lambda f(x+1) + \mu(w(x)(f(0)+c_s) + (1-w(x))f(x)))
\]

**Theorem**

Let \( U_l \) be the value function for threshold \( l \) policy. If

\[
U_l(l - 1) \leq U_l(0) + c_s < U_l(l),
\]

then \( U_l \) is quasiconvex

**Definition**

A function \( f \) on \( Z_+ \) is quasiconvex (unimin) if \( f(x + 1) - f(x) \geq 0 \)

for all \( x > y \) whenever \( f(y + 1) - f(y) > 0 \).
Proof of Threshold Optimality

• **Suppose** we could find an $l^*$ for which:
  1. $U_{l^*}$ is quasiconvex
  2. $U_{l^*}(l^* - 1) \leq U_{l^*}(0) + c_s < U_{l^*}(l^*)$

• This implies:

  $U_{l^*}(x) \begin{cases} 
  \leq U_{l^*}(0) + c_s & \text{if } x \leq l^* - 1 \\
  > U_{l^*}(0) + c_s & \text{if } x \geq l^* 
  \end{cases}$

• Then, $f(x) = U_{l^*}(x)$ is a solution to the fixed point equation for optimal DP operator:

  $Tf(x) = c(x) + \beta \{ \lambda f(x+1) + \mu \min(f(x), f(0) + c_s) \}$

But we only need to look for an $l^*$ for which condition (2) holds since (2) $\Rightarrow$ (1), by theorem.
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Proof of Threshold Optimality

**Lemma**

\[ l^* = \min \{ l : U_l(l) > U_l(0) + c_s \} \text{ satisfies } (2) \]

Proof: Suppose not. \( U_{l^*}(l^* - 1) > U_{l^*}(0) + c_s \). Then:
Proof of Threshold Optimality

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Proof: Suppose not. \( U_{l^*}(l^* - 1) > U_{l^*}(0) + c_s \). Then:

- Policy iteration on decision at \( l^* - 1 \) \( \Rightarrow \) threshold \( l^* - 1 \) strictly improves threshold \( l^* \) policy.
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- which would be a contradiction, unless:
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- \( \ldots \) which contradicts definition of \( l^* \)
Two queues

- Assume cost, \( c(x_1, x_2) \) is monotone and has no service costs
- \( n \) step value function \( V_n \) is recursively given by:

\[
V_{n+1}(x_1, x_2) = c(x_1, x_2) \beta \lambda_1 V_n(x_1 + 1, x_2) + \lambda_2 V_n(x_1, x_2 + 1) + \\
\mu \min(V_n(x_1, 0), V_n(0, x_2), V_n(x_1, x_2))
\]
Lemma

$V_n$ is increasing. i.e., if $(x_1, x_2)$ and $(y_1, y_2)$ are such that $x_1 \leq y_1$ and $x_2 \leq y_2$ then $V_n(x_1, x_2) \leq V_n(y_1, y_2)$

The optimal control $u_n$ is:

$$u_n(x_1, x_2) = \begin{cases} 
1, & \text{if } V_n(x_1, 0) \leq V_n(0, x_2) \\
2, & \text{otherwise.}
\end{cases}$$
The optimal control with $n$ steps to go is given by a switch curve:

$$u_n(x_1, x_2) = \begin{cases} 
1, & \text{if } x_2 \geq s_n(x_1) \\
2, & \text{otherwise.}
\end{cases}$$

where

$$s_n(x) = \min\{y : V_n(x, 0) \leq V_n(0, y)\}$$
Further Work: The general problem for $n > 2$ queues

- An index rule is given by $n$ functions $\psi_1, \ldots, \psi_n$ such that the control is given as:

$$u(x_1, \ldots, x_n) = \arg \max_{i \in [n]} \{ \psi_i(x_i) \}$$

- Can the optimal control be described by index rules?

- Approximate algorithms using index policies
  - Longest queue scheduling corresponds to $\psi_i(x) = x$
  - LWF scheduling, which has been found to be ‘competitive’ in CS literature corresponds to using an index rule where:

$$\psi_i(x) = \frac{x}{\sqrt{\lambda_i}}$$
Thank you!